केन्द्रीय विद्यालय संगठन

KENDRIYA VIDYALAYA SANGATHAN

REGIONAL OFFICE, CHENNAI

2 DAY WORKSHOP FOR PGT(MATHEMATICS

16.08.22 & 17.08.22

VENUE

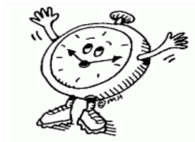
KENDRIYA VIDYALAYA, ASHOK NAGAR, CHENNAI

REFERENCE MATERIAL

INDEX

KENDRIYA VIDYALAYA SANGATHAN, REGIONAL OFFICE, CHENNAI

02 DAY WORKSHOP FOR PGT(MATHS) : 16.8.2022 & 17.8.2022



DAILY SCHEDULE

DAY / DATE	I SESSION 9.00 A.M. TO 11.00 A.M.	II SESSION 11.15 A.M. TO 1.00 P.M.	III SESSION 2.00 P.M. TO 3.45 P.M.	IV SESSION 4.00 P.M. TO 5.30 P.M.
16.7.2022	Inauguration Curriculum 2022-2023	Misconceptions and Error Analysis	Tools to make concepts clear	Group work on Error Analysis
	Sh Sankara Subramaian	Sh M Srinivasan	Sh M Srinivasan	
17.8.2022	Session on Creative and Critical Thinking	Math Phobia, Action plan for Meticulous Revision	Presentation of Group Work	Presentation of Group Work Valedictory Function
	Sh M Srinivasan	Sh Sankara Subramaian		

Tea Time: 10.45 A.M. TO 11.00 A.M. & 03.45 P.M. TO 04.00 P.M.

Lunch: 1.00 P.M. TO 02.00 P.M.

DAILY REPORT

DAY 1 : 16.08.2022

The 2 day workshop PGT(Maths) started with Prayer and lighting of lamp by Ms T Rukmani, Officiating Deputy Commissioner, KVS, RO, Chennai

After self-introduction by the participants, Sh Om Prakash, Principal, Kendriya Vidyalaya, Ashok Nagar, Chennai welcomed all the participants.

In her inaugural speech, Ms T Rukmani, Officiating Deputy Commissioner, KVS, RO, Chennai emphasized the need of clarity in teaching Mathematics to the students from lower classes. She also listed out the difficulties faced by the students and teachers during the teaching and learning process of Mathematics. She instructed the participants to compile material which can be used by all teachers handing class XII Mathematics.

Resource Person Sh M.Srinivasan, PGT (Maths), Kendriya Vidyalaya, Ashok Nagar Chennai explained the objective of the workshop and the tasks to be completed by participants .

Resource Person Sh Sankara Subramanian, PGT(Maths), Kendriya Vidyalaya, No.1 Tambaram, Chennai discussed the Mathematics Curriculum 2022-2023 in detail. He explained the topics deleted and also the 'topics deleted but not to be ignored'.

After tea break, Resource Person Sh M.Srinivasan took a session on 'Misconception in Mathematics'. He explained the difference between 'Mistake and Error' and different types of error. He also explained various types of errors with examples. Errors in selected topics were discussed. The template to be used for Error Analysis in the topics allotted to the participants was shared and discussed.

After lunch break, Resource Person Sh M.Srinivasan took a session on 'Tools to make concepts clear'. He explained how the tools like 'Geogebra, live worksheets, Autograph, Edpuzzle' can be used to make concepts clear. Worksheets on selected topics using the tools were discussed.

After tea break, the participants continued their group work of compiling the task assigned to them.

DAY 2 : 17.08.2022

The second day of the workshop started with the morning prayer and recollection of topics discussed on I day.

Resource Person Sh M.Srinivasan took a session on 'Critical and Creative Thinking'. He explained the importance of critical and creative thinking and how CCT helps in problem solving. Problems on CCT was also explained.

Resource Person Sh Sankara Subramanian took a session on 'Meticulous revision and Math Phobia'. He explained the strategies to be followed for effective revision and discussed method to be followed. He also how Math Phobia among students can be tackled and discussed some tips for the same.

After tea break and lunch break the participants continued their group work and compiled the work for presentation.

The material prepared and compiled by the participants were presented after evening tea break. Ms T Rukmani, Officiating Deputy Commissioner, KVS, RO, Chennai gave her suggestions for the improvement of the material prepared.

In her valedictory address, Ms T Rukmani, Officiating Deputy Commissioner, KVS, RO, Chennai instructed the participants to give more emphasis on concept clarity while teaching and meticulous training. She also instructed the teachers to concentrate on slow bloomers and strive hard for qualitative and quantitative result.

CHAPTER I : RELATIONS & FUNCTIONS

CONCEPT MAPPING

1. RELATIONS AND FUNCTIONS

Cartesian Product:
The Cartesian product of two sets A and B = $A \times B = \{(a, b): a \in A, b \in B\}$

Relation: – A relation R is a subset of the Cartesian product $A \times B$.
Domain : – The set of all first elements of the ordered pairs in a relation R from a set A to set B is called the domain of the relation R.

Range : – The set of all second elements in a relation R from a set A to set B is called the range of the relation R. The set B is called the co domain of the relation R.
If $(a, b) \in R$ we say that 'a is related to b' under the relation R and we write as a R b.

Ex. :– Let A={3,4} and B={1,6}. Find R such that $\{(a, b): a < b, a \in A, b \in B\}$ and also its domain and range.
Solution: R= {(3, 1),(4,1)}, Domain of R= {3,4}, Range of R={1}
NOTE: Number of relations from set A to set B having n and m elements = 2^{nm}
<u>Types of Relations</u>
1. <u>Empty Relation</u> :– A relation R in a set A is called empty relation if $R = \phi$
2. <u>Universal Relation</u>:– A relation R is universal relation, if $R = A \times A$
3. Reflexive Relation:– A relation R in a set A is called reflexive,
If $(a,a) \in R$, for every $a \in A$.
4. Symmetric Relation:– A relation R in a set A is called symmetric,
if$(a,b) \in R \ \in R \Rightarrow (b,a) \in R$, $\forall$ $(a,b) \in R \in A$.
5. Transitive Relation:– A relation R is called transitive,
if $(a,b) \in R$, $(b,c) \in R$ implies that$(a,c) \in R$, $\forall$ $a,b,c \in A$.
6. Equivalence Relation:– A relation R is said to be an equivalence relation , if R is reflexive, symmetric and transitive.
Example:– Let P be the set of all lines in a plane, and the relation R in set P is given by $R = \{(a,b) \in P \times P$: line a is parallel to line b$\}$ show that R is an equivalence relation.

Example:– Let T be the set of all triangles in a plane with R is a relation in T is given by R={T_1,T_2}: T_1 is congurent to T_2} show that R is an equivalence relation.
<u>Example</u> : In N $\times$ N, show that the relation defined by (a,b)R(c,d) iff ad=bc is an equivalence relation

<u>FUNCTIONS</u>

Function:– A relation f from a set A to set B is said to be a function if every element of Set A has one and only one image in Set B.
The function f from A to B is denoted by f: A→B

Types of Functions.

One–One (or injective) function A function f:X→Y is defined to be one–one (or injective) if $x_1, x_2 \in X$, $f(x_1)=f(x_2) \Rightarrow x_1=x_2$ or If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ $\forall x_1, x_2 \in X$
Onto (or surjective)function: A function f: X → Y is said to be onto, if every element of Y is the image of some element of X under f, i.e for every y$\in$ Y, there exists an element x in X s.t f(x)=y see fig 1 (iii) and (iv)

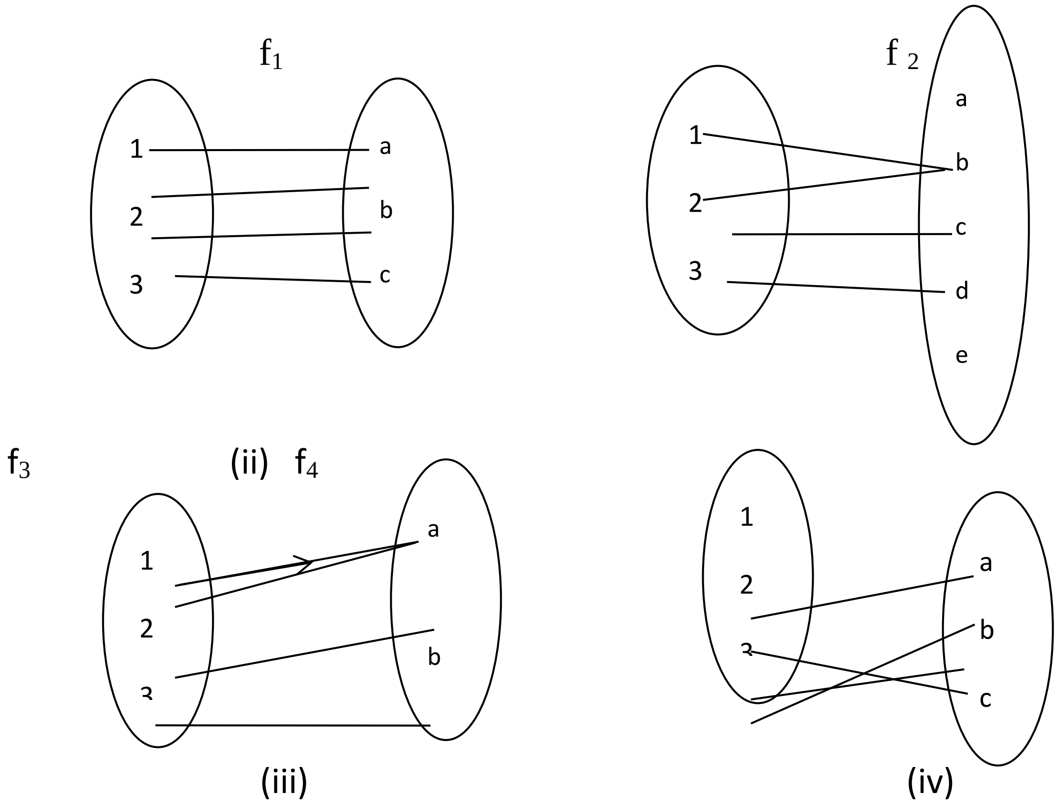
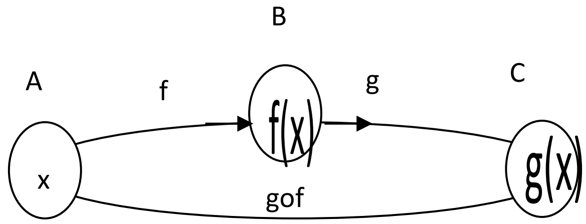


Fig 1 (i) to (iv)

Composition of functions:– let f : A→ B and g : B→C be two functions. Then the composition of f and g, denoted by gof, is defined as the function gof : A→C given by gof (x) = g(f (x)) $\forall$ x $\in$A

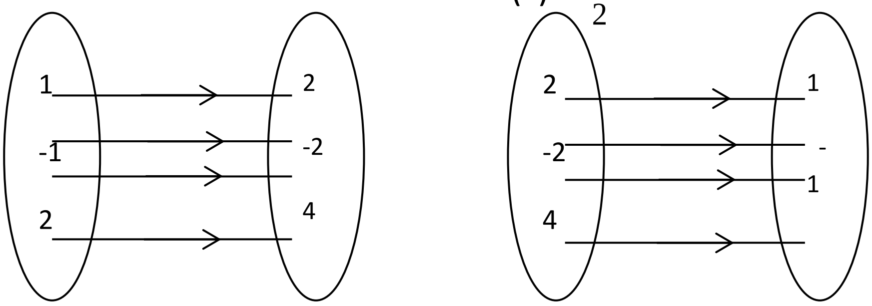


Note: gof: A→C is one–one and f: A→B is onto⇒g: B→C is one –one.

Invertible function

Def: – Let f: A→B be a bijection, Then a function g: B→A which associates each element y$\in$B to a unique element x$\in$A s.t. f(x) = y is called the Inverse of f i.e f(x)= y ⇒g(y) =x. The Inverse of f is generally denoted by f^{-1}

A f(x) =2x B B $f^{-1}(x) = \frac{1}{2}x$ A



QUESTIONS FOR PRACTICE : RELATIONS AND FUNCTIONS

LEVEL 1

1)Let T be the set of all triangles in a plane with R a relation in T given by

R=$\{(T_1, T_2): T_1 is\ congruent\ to T_2\}$ .Show that R is an equivalence relation

2)Let L be the set of all lines in XY plane and R be the relation in L defined as R=$\{(L_1, L_2): is\ parallel\ to L_2\}$ .Show that R is an equivalence relation

3)Show that the relation R in the set $\{1,2,3\}$ given by R =$\{(1,2), (2,1)\}$ is symmetric but neither reflexive nor transitive

4)Determine whether the relation R in the set A=$\{1,2,3,4,5,6,7 \ldots \ldots \ldots 13,14\}$defined as R=$\{(x, y): 3x - y = 0\}$ is reflexive, symmetric and transitive

5) Show that the relation R in R defined as R= $\{(a, b): a \leq b\}$,is reflexive and transitive but not symmetric

Level 2

1) Show that the relation R in the set of real numbers defined as R=$\{(a, b): a \leq b^2\}$ is neither reflexive ,nor symmetric , nor transitive .
2) Show that the relation R in the set A=$\{x \in Z: 0 \leq x \leq 12\}$ given by R=$\{(a, b): |a - b| is\ a\ multiple\ of\ 4\}$ is an equivalence relation.
3) Show that the relation R defined by $(a, b)R(c, d) \Rightarrow a + d = b + c$ on A x A where A=$\{1,2,3, \ldots \ldots \ldots \ldots 10\}$is an equivalence relation.

4)Determine whether the following relation is reflexive ,symmetric and transitive Relation R in the set N of natural numbers defined R=$\{(x, y): y = x + 5 and\ x < 4\}$

5.Check whether the relation R defined on the set A=$\{1,2,3,4,5,6\}$ as

R= $\{(a, b): b = a + 1\}$ Is reflexive, symmetric or transitive

Level 3

1)Let R be the relation on NxN defined by (a,b)R(c,d)⇔ad(b+c)=bc(a+d)

Check whether R is an equivalence relation on NxN

2)Let A={1,2,3}.Find the number of equivalence relations on A containing (1,2)

3) Let A={1,2,3} . Find the number of relations on A containing (2,1) and (2,3) which are reflexive and symmetric but not transitive

4.Determine whether the relation R defined on the set R of real numbers as

R $=\{(a, b): a, b \in R \quad and\ a-b+\sqrt{3} \ \in S, where\ S\ is\ the\ set\ of\ all\ irrational\ numbers\}$ is reflexive ,symmetric and transitive

5.Let A be the set of all positive integers and R be a relation on AxA defined by (a,b)R(c,d)⇔ ad=bc for all (a,b), (c,d) in AxA . Show that R is an equivalence relation on AxA

6. Show that the function f : $R_o \to R_o$, defined as f(x) = 1/x, is one–one onto where R_o is the set of all non-zero real numbers.

7. Check the surjectivity of the function f : Z → Z given by f(x) =|x|.

8) Let A= R–{2} and B=R–{1} If f: A→B is a function defined by f(x)= (x–1 / x-2) show that f is a bijection.

9) . Let $f: N \to N$ be defined by $f(x)=\begin{cases}\dfrac{n+1}{2}, if\ n\ is\ odd\\ \dfrac{n}{2}, if\ n\ is\ even\end{cases}$

Examine whether f is one to one, onto or bijective, Justify your answer.

10) Let A and B be sets. Show that $f: A\times B \to B\times A$, Such that $f(a,b)=(b,a)$ is bijective function.

11) State whether the function $f: R\to R$ defined by $f(x)=1+x^2$ is one-one, onto or bijective .

12) Prove that the function f : N × N defined by f(x) = x^2+x+1 is one one but not onto.

ERROR ANALYSIS : RELATIONS AND FUNCTIONS

Q.1 Check whether the relation R in R= {(a,b):a$\leq$ $\boldsymbol{b^2}$ $\boldsymbol{where\ a, b} \in \boldsymbol{R}$} is reflexive.

Ans. It is reflexive as $2 \leq \mathbf{2^2}$ and $3 \leq \mathbf{3^2}$ where $2,3 \in \boldsymbol{R}$

(Misconception student does not think proper counter example)

Right ans. R is not reflexive as $\frac{1}{2} \geq (\frac{1}{2})^2$ where $\frac{1}{2} \in \boldsymbol{R}$

Explanation: relation will be reflexive when it satisfy the condition for all elements of given set

Q.2 Check whether the relation R in R, R= {(a,b):a$\leq$ $\boldsymbol{b^2}$ $\boldsymbol{where\ a, b} \in \boldsymbol{R}$} is symmetric

Answer: It is symmetric as $2 \leq \mathbf{3^2}$ and $3 \leq \mathbf{2^2}$ so $(2,3),(3,2) \in \boldsymbol{R}$.

(Misconception student does not think proper counter example)

Right answer : R is not symmetric as $\mathbf{2} \ \leq \mathbf{9^2}\ \boldsymbol{but}\ \mathbf{9} \geq \mathbf{2^2}$ so $(2,9) \in \boldsymbol{R}$ but $(9,2) \notin \boldsymbol{R}$

Explanation: relation will be symmetric when it satisfy the condition for all elements of given set where (a,b) $\in \boldsymbol{R}$ as well as (b,a) $\in \boldsymbol{R}$

3. Let A = { 1,2,3}. Check whether R = { (1,2),(2,1), (1,1), (1,3)} is symmetric or not.

Mistake done : Here, $(1,2) \in R$, (2,1))$\in R$. So, it is symmetric.

Correction : The student thinks that only an ordered pair is to check for Symmetric. So, he forgets to check for (1,3).

4. If A = {1,2,3}, check whether R = {(1,1),(1,2),(2,1)} is transitive or not.

Mistake done: (1,2))$\in R$, (2,1))$\in R$ implies that (1,1))$\in R$. So it is transitive.

Correction: Here the student forgets to see for (2,1))$\in R$, (1,2))$\in R$ implies (2,2))$\in$ Ror not.

5.. Which of the following are functions from A = {1,2} to B = { a,b} Find f^{-1} (if exists).

(i). f = { (1,a), (2,a)}

(ii) f= { (1,a),(2,b),(1,b)}

(i). **Mistake done:** f^{-1} = { (a,1),(a,2)}.

Correction : The child got confusion for finding the inverse of a relation and that of a function. He forgot to see for one-one onto.

(ii). **Mistake done:** f^{-1} = { (a,1),(b,2),(b,1)}.

Correction : The child got confusion for finding the inverse of a relation and that of a function. He forgot to see for one-one onto.

6.Mistake done : Students use example to prove a relation which is reflexive or symmetric or transitive.

Correction : When we prove any result it must be generally but when we disprove any result_then give example.

7. R= {(1,1),(2,2),(3,3)} Is R transitive relation?

Mistake done: R is not transitive.

Correction: R is transitive since we do not have any two elements (a,b) , (b,c) ∈ R such that (a,c)∈ R.

MISCONCEPTIONS	STRATEGY FOR ITS REMOVAL
If R= {(a,b):a$\leq \boldsymbol{b^2}$ $\boldsymbol{where\ a, b} \in \boldsymbol{R}$} is reflexive as 2 $\leq \mathbf{2^2}$	R is not reflexive as $\frac{1}{2} \geq (\frac{1}{2})^2$ where $\frac{1}{2} \in \boldsymbol{R}$
If R= {(a,b):a$\leq \boldsymbol{b^2}$ $\boldsymbol{where\ a, b} \in \boldsymbol{R}$} is symmetric as 2 $\leq \mathbf{3^2}$ and 3$\leq \mathbf{2^2}$ so (2,3),(3,2) $\in \boldsymbol{R}$	R is not symmetric as $\mathbf{2} \leq \mathbf{9^2}$ $\boldsymbol{but}$ $\mathbf{9} \geq \mathbf{2^2}$ so (2,9)$\in \boldsymbol{R}$ but (9,2)$\notin \boldsymbol{R}$

INVERSE TRIGONOMETRIC FUNCTIONS

BASIC CONCEPTS

In general trigonometric functions are not one – one functions

sin 30° =1/2 , sin150° = 1/2

$\therefore$ **Trigonometric functions are not one – one functions**

But by restricting the domain inverse trigonometric functions can be defined

I – Domain and Range of Inverse Trignometric functions:

The value of an Inverse trignometric function which lies in the range of Principal branch is called the Principal value of the Inverse trignometric function.

Example:

1. The Principal value of $\sin^{-1}(\sqrt{3}/2) = \pi/3$
2. The Principal value of $\tan^{-1}(-1) = -\pi/4$

Properties of Inverse Trigonmetric Functions

	Function	*Domain*	*Range (Principal Value Branch)*
1	$\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
2	$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
3	$cosec^{-1}x$	$R-(-1,1)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$
4	$\sec^{-1}x$	$R-(-1,1)$	$[0,\pi]-\{0\}$
5	$\tan^{-1}x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
6	$\cot^{-1}x$	R	$(0,\pi)$

II.

$$\boxed{\sin^{-1}(\sin x) = x}$$

$cos^{-1}\,(\,cos\,x\,) = x$
$tan^{-1}\,(\,tan\,x\,) = x$
$cosec^{-1}\,(\,cosec\,x\,) = x$
$sec^{-1}\,(\,sec\,x\,) = x$
$cot^{-1}\,(\,cot\,x\,) = x$

III.

$sin^{-1}\,(-x\,) = -\,sin^{-1}\,(x\,)$
$cosec^{-1}\,(-x\,) = -\,cosec^{-1}\,(x\,)$
$tan^{-1}\,(-x\,) = -\,tan^{-1}\,(x\,)$
$cos^{-1}\,(-x\,) = \pi -\ cos^{-1}\,(x\,)$
$(\,sec\,x\,) = \pi -\ sec^{-1}\,(x\,)$
$cot^{-1}\,(\,cot\,x\,) = \pi -\ cot^{-1}\,(x\,)$

IV.

$$sin^{-1}\left(\frac{1}{x}\right) = cosec^{\,-1}\,(x\,)$$

$$cosec^{-1}\left(\frac{1}{x}\right) = sin^{\,-1}\,(x\,)$$

$$cos^{-1}\left(\frac{1}{x}\right) = \ sec^{-1}\,(x\,)$$

$$sec^{-1}\left(\frac{1}{x}\right) = \ cos^{-1}\,(x\,)$$

$$tan^{-1}\left(\frac{1}{x}\right) = cot^{\,-1}\,(x\,)$$

$$\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(x)$$

V.

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$$

VI.

$$\sin^{-1} x = \cos^{-1} \sqrt{1 - x^2}$$

$$\cos^{-1} x = \sin^{-1} \sqrt{1 - x^2}$$

$$3 \sin^{-1} x = \sin^{-1}(3x - 4x^3)$$

$$3 \cos^{-1} x = \cos^{-1}(4x^3 - 3x)$$

VII.

$\sin^{-1} x = \cos^{-1} \sqrt{1 - x^2}$
$\cos^{-1} x = \sin^{-1} \sqrt{1 - x^2}$
$3 \sin^{-1} x = \sin^{-1}(3x - 4x^3)$
$3 \cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

VIII.

$$2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$

$$2 \tan^{-1} x = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$$

$$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$$

VIII.

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$$

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right)$$

$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(x\,y - \sqrt{1 - x^2}\sqrt{1 - y^2}\right)$$

QUESTION BANK INVERSE TRIGONOMETRIC FUNCTIONS

LEVEL-1

1. Find the Principal value of $\sin^{-1} \sqrt{3}/2$
2. Find the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.
3. Evaluate $\cos\left\{\pi/3 - \cos^{-1}(1/2)\right\}$

LEVEL - 2

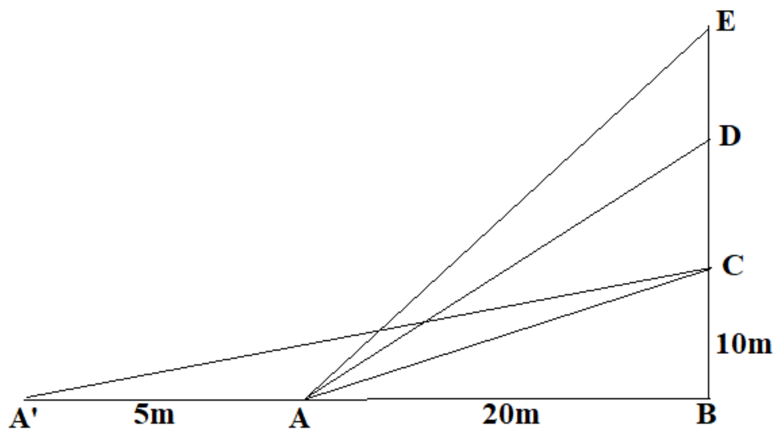
1. Write the domain and Principal branch of $\tan^{-1} x$.
2. Find the principal value of : $\sin^{-1}\left(-\frac{1}{2}\right)$

LEVEL-3

1. Give an example of a relation on A={1,2,3,4}, which is Symmetric but neither
2. Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.
3. Find the value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.
4. Find the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.
5. Find the value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$.

<u>**CCT INVERSE TRIGONOMETRIC FUNCTION**</u>

The Government of India is planning to fix a hoarding board at the face of a building on the road of a busy market for awareness on COVID-19 protocol. Ram, Robert and Rahim are the three engineers who are working on this project. "A" is considered to be a person viewing the hoarding board 20 metres away from the building, standing at the edge of a pathway nearby. Ram, Robert and Rahim suggested to the firm to place the hoarding board at three different locations namely C, D and E. "C" is at the height of 10 metres from the ground level. For the viewer A, the angle of elevation of "D" is double the angle of elevation of "C" The angle of elevation of "E" is triple the angle of elevation of "C" for the same viewer. Look at the figure given and based on the above information answer the following:



1. Measure of $\angle CAB$ =

a. $\tan^{-1}(2)$

b. $\tan^{-1}\left(\frac{1}{2}\right)$

c. $\tan^{-1}(1)$

d. $\tan^{-1}(3)$

2. $Measure\ of \angle DAB\ =$

a. $\tan^{-1}\left(\frac{3}{4}\right)$

b. $\tan^{-1}(3)$

c. $\tan^{-1}\left(\frac{4}{3}\right)$

d. $\tan^{-1}(4)$

3. $Measure\ of\ \angle EAB\ =$

a. $\tan^{-1}(11)$

b. $\tan^{-1} 3$

c. $\tan^{-1}\left(\frac{2}{11}\right)$

d. $\tan^{-1}\left(\frac{11}{2}\right)$

4. $A^{|}$ Is another viewer standing on the same line of observation across the road. If the width of the road is 5 meters, then the difference between $\angle CAB$ and $\angle CA'B$ Is

a. $\tan^{-1}(1/2)$

b. $\tan^{-1}$ (1/8)

c. $\tan^{-1}\left(\frac{2}{5}\right)$

d. $\tan^{-1}\left(\frac{11}{21}\right)$

5. Domain and Range of $\tan^{-1} x =$

a. R^{+}, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b. R^{-}, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

c. R , $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

d. R , $\left(0\ , \frac{\pi}{2}\right)$

Common Mistakes

1. **Mistake done :** $\cos^{-1}x = \frac{1}{cosx}$

Correction: $\cos^{-1}x$ is inverse function of cosx. It is not reciprocal of cosx.

Correction: R is transitive since we do not have any two elements (a,b) , (b,c) ∈ R such that (a,c)∈ R.

2. Identification of Principal Values

MATRICES AND DETERMINANTS

CONCEPT MAPPING

1. What is matrix?

* Matrix is a rectangular array of objects.

* $\begin{bmatrix} 3 & 4 & 6 \\ 1 & 0 & 7 \end{bmatrix}$ ←Rows

↑ Columns

2. Order of matrix.

* If a matrix has ‘m’ number of rows and ‘n’ number of columns then order of matrix is mxn.

* $\begin{bmatrix} 6 & 9 \\ 4 & 3 \end{bmatrix}$ order -2x2 i.e. A square matrix of order 2

* $\begin{bmatrix} 7 & 1 & 3 \\ 5 & 7 & 9 \end{bmatrix}$ order 2x3 i. e. A rectangular matrix of order 2x3.

3. Types of matrices.

* Row matrix--- $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

* Column matrix ---$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

* Square matrix of order 2 ---$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, of order 3----$\begin{bmatrix} 9 & 7 & 2 \\ 1 & 6 & 3 \\ 5 & 0 & 6 \end{bmatrix}$

*Diagonal matrix --$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 7 \end{bmatrix}$,in which all the nom diagonal elements of a square matrix are all zero

*Scalar matrix --$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$(it is a diagonal matrix whose all diagonal elements are equal)

*Unit/Identity matrix—I_2= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

* Null/Zero matrix --$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4. Equality of matrices.

5. Operation on matrices

(a) Addition/subtraction of matrices.

* Same order matrices can be added as well as subtracted.

* Addition as well as subtraction of matrices are binary operations.

(b) Multiplication of matrices.

*Two matrices are compatible for multiplication if number of columns of first matrix is equal to the number of rows of second matrix.

*$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 7 & 4 \\ 2 & 5 & 2 \end{bmatrix}$, AB exists but BA does not exist.

* Multiplication of matrices is also binary operation under certain conditions.

6.. Transpose of matrix.

* By interchanging rows to columns or vice versa transpose of matrix is obtained.

*$A = \begin{bmatrix} 2 & 5 & 8 \\ 1 & 7 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 1 \\ 5 & 7 \\ 8 & 0 \end{bmatrix}$

7. Symmetric and skew symmetric matrices.

(a) Symmetric matrix if $A=A^{T}$,(where A is a square matrix)

(b) Skew symmetric matrix if $A^T= -A^T$

(c) Every square matrix can be expressed as the sum of a symmetric and skew symmetric matrices.

i.e. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

=(Symmetric matrix)+ (Skew symmetric matrix)

8. Elementary transformations for finding inverse of a matrix.

(a) Elementary row transformations A=IA

(b)Elementary column transformations A=AI

Note: Elementary row transformations and Elementary column transformations should not be used simultaneously to find the inverse of a matrix.

9.What is determinant?

*Every square matrix can be associated with a unique number that is called its determinant.

i.e.$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$=ad-bc

10. Minors and co factors of the elements of a determinant.

* $A_{ij}=(-1)^{i+j}M_{ij}$,where A_{ij} is co factor and M_{ij} is minor.

11. Evaluation of a determinant.

* The sum of the products of elements and its corresponding co factors of a row or column.

12. Properties of determinants.

* The value of the determinant remains unchanged if its rows and columns are interchanged.

*If any two rows(or columns) of a determinant are interchanged the sign of determinant changes.

* If any two rows (or columns) of a determinant are identical(all corresponding elements are same) then value of determinant is zero.

* If each element of a row (or a column) of a determinant is multiplied by a constant 'k' then its value gets multiplied by' k'

*If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

*If ,to each element of any row or column of a determinant , the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same.

* Use of appropriate property to evaluate the determinant easily.

13. Adjoint and inverse of a matrix.

*adjA=$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T=\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$,where A_{ij} is a co factor of a_{ij}.

* A(adj A)=I |A|=(adjA)A

$\frac{*\quad A\,(adjA)}{|A|}$=I=$\frac{(adj)A}{|A|}$

*$A^{-1}=\frac{adjA}{|A|}$

14. Conditions for a system of linear equations to be consistent/inconsistent

* A system of linear equations is called consistent if it has at least one solution.

i) $|A| \neq 0$ (System has unique solution)

ii) $|A| = 0 \neq (adj\ A)B$ (System has no solution)

iii) $|A| = 0 = (adj\ A)B$ (System has infinitely many solutions)

15. Solving a system of linear equations using matrix method.

3x+2y-5z=10

2x+7y-4z =20

8x+6y-5z=25

A $=\begin{bmatrix} 3 & 2 & -5 \\ 2 & 7 & -4 \\ 8 & 6 & -5 \end{bmatrix}$, X=$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, B=$\begin{bmatrix} 10 \\ 20 \\ 25 \end{bmatrix}$

AX=B, X=A^{-1}B

16.To form linear equations from given word problems and their solution using matrix method.

Matrices

S.NO	QUESTIONS
	LEVEL – 1 (EASY)
1	If A = $\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 4 \end{bmatrix}$, find whether BA exist. If exist, then write the order.
2	If A = $\begin{bmatrix} -1 & 4 \\ 1 & 3 \end{bmatrix}$ and B^T = $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, then find $(7A + 5B)^T$.
3	Write a 3 × 2 matrix A, whose elements are given by $a_{ij} = \frac{1}{3}\left\|-3i + j\right\|$.
4	If A = [a_{ij}] = $\begin{bmatrix} 2 & 3 & -5 \\ 1 & 4 & 9 \\ 0 & 7 & -2 \end{bmatrix}$ and B = [b_{ij}] = $\begin{bmatrix} 2 & 1 & -1 \\ -3 & 4 & 4 \\ 1 & 5 & 2 \end{bmatrix}$, then find $a_{22} + b_{21}$
5	Given A $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ = $\begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$, write the order of matrix A.
	LEVEL – 2 (AVERAGE)
1	Express the matrix A = $\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$ as sum of a symmetric and a skew symmetric matrix.
2	Show that all the diagonal elements of a skew symmetric matrix are zeros.
3	If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, then find x and y such that $A^2 - xA + yI = 0$.
4	Write the value of x – y + z from the following equation: $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$
5	For A = $\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$.
	LEVEL – 3 (DIFFICULT)
1	If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{pmatrix}$ is a matrix satisfying $AA^T = 9I$ where A^T denotes the transpose of A, then find the value of \`$x$\`.

2	There are 3 families A , B and C. The number of men, women and children in theses families are as under: <table><tr><td></td><td>Men</td><td>Women</td><td>Children</td></tr><tr><td>Family A</td><td>2</td><td>3</td><td>1</td></tr><tr><td>Family B</td><td>2</td><td>1</td><td>3</td></tr><tr><td>Family C</td><td>4</td><td>2</td><td>6</td></tr></table>Daily expenses of men , women and children are Rs.200, Rs.150 and Rs.200 respectively. Only men and women earn and children do not. Using matrix multiplication, calculate the daily expenses of each family.
3	To raise money for an orphanage, students of three schools A , B and C organized an exhibition in their locality, where they sold paper bags, scrap-books and pastel sheets made by them using recycled paper, at the rate of Rs.20, Rs.15 and Rs.5 per unit respectively. School A sold 25 paper bags, 12 scrap books and 34 pastel sheets. School B sold 22 paper bags, 15 scrap books and 28 pastel sheets while School C sold 26 paper bags, 18 scrap books and 36 pastel books. Using matrices, find the total amount raised by each school.
4	Let A = $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, B = $\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$ and C = $\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find a matrix D such that C D – A B = 0.
5	If X $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ = $\begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$, then find the matrix X.

Application of matrices

LEVEL I QUESTIONS

1	System of linear equations are consistent with unique solutions If ______________________
2	System of linear equations are consistent with infinite number of solutions if________________
3	System of linear equations are in consistent If ______________________

4	If A is an invertible square matrix of order 2 and A^2 -A +2! = 0 then find A^{-1}
5	Solve by matrix method $\frac{1}{x}+\frac{1}{y}$ = 3 and $\frac{1}{x}-\frac{1}{y}$ = 1

LEVEL II QUESTIONS

1	Solve by matrix method 2x+3y = 5 3x – 2y =1
2	Examine the the consistency of the equations 2x – y = 5 and x + y = 4
3	If the system of equations is consistent find k 2x +ky =3 5x-3y = 2
4	Examine the the consistency of the equations x + 3y and 2x + 6y = 8
5	Examine the the consistency of the equations x + y + z =1; 2x + 3y + 2z = 2 and ax + ay + 2az = 4

LEVEL III QUESTIONS

1	If A=$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1} and hence solve the system of equations 2x-3y+5z = 16 , 3x+2y-4z =- 4, x+y-2z = -3

2	If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Hence Solve the system of equations; $x - 2y = 10$ $2x - y - z = 8$ $-2y + z = 7$
3	Evaluate the product AB, where $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ *Hence solve the system of linear equations* x-y=3 , 2x+3y+4z=17 , y+2z = 7
4	Solve $x - y + 2z = 7$ $3x + 4y - 5z = -5$ $2x - y + 3z = 12$
5	. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs. 60. The cost of 2 kg onion, 4 kg wheat and 2 kg rice is Rs. 90. The cost of 6 kg onion, 2 k wheat and 3 kg rice is Rs. 70. Find cost of each item per kg by matrix method.

TOPIC : DETERMINANTS AND APPLICATION OF DETERMINANTS

LEVEL – 1

1. **Evaluate ,** $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$
2. **Find ,** $\begin{vmatrix} sin10° & -cos10° \\ sin80° & cos80° \end{vmatrix}$
3. **Find the equation of the line joining points (1,1) and (2,2) using determinants.**
4. **Using determinants , show that the points (11,7),(5,5) and (-1,3) are collinear.**
5. **Find the value of x for which** $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix}$ = $\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

Level – 2

6. **Find the roots of the equation** $\begin{vmatrix} 0 & x & 16 \\ x & 5 & 7 \\ 0 & 9 & x \end{vmatrix}$ **= 0**
7. **Find** $\begin{vmatrix} a + ib & c + id \\ -c + id & a - ib \end{vmatrix}$
8. **Find the maximum value of** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + sin\emptyset & 1 \\ 1 + cos\emptyset & 1 & 1 \end{vmatrix}$
9. **If the points (a, b) (a^1 , b^1) and (a-a^1 , b-b^1) are collinear , show that ab^1 =a^1b.**
10. **Find the area of triangle , whose vertices are (a, a+ c) , (b, c+ a) and (c, a+b).**

Level – 3

11. **Find the equation of the line joining A(9, 1,3) and B(0,0) using determinants and find k if D(k,0) is a point such that area of triangle ABD is 3 square units.**
12. **If the matrix** $\begin{pmatrix} 1 & 3 & \alpha + 2 \\ 2 & 4 & 8 \\ 3 & 5 & 10 \end{pmatrix}$ **is singular , find** α
13. **If x,y,z ∈ N , then find** $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

14.Prove that $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ **is independent of** θ**.**

15.**If x= -4 is a root of** $\begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$ **= 0 , find other two roots**

TOPIC: ADJOINT AND INVERSE OF A MATRIX (CHAPTER- 4)

LEVEL : I

1. **Find the matrix A , if** $\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} A \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 3 & -2 \end{pmatrix}$**.**

   Ans: A = $\frac{-1}{4}\begin{pmatrix} -53 & 18 \\ 25 & -10 \end{pmatrix}$

2. **If A =** $\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}$ **be such that** $A^{-1} = kA$**, then find the value of k.**

   (Ans : k = 1/19)

3. **If A =** $\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ $and\ B = \begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix}$**, verify that** $(AB)^{-1} = B^{-1}A^{-1}$**.**

4. **Find the inverse of the matrix A =** $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ **, and hence show that** $A^{-1}A = I$ **.**

5. **If A =** $\begin{pmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{pmatrix}$ **, find adjoint of A, and hence verify that A (adj A) =** $|A|\, I_n$**.**

LEVEL: II

6. **Show that A =** $\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}$ **satisfies the equation** $x^2 - 6x + 17 = 0$**. Hence find** A^{-1}**.**

7. **Find the matrix A satisfying the matrix eqn:** $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} A \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

8. **X is an unknown matrix satisying the eqn.** **:** $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} X = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ **. Using concept of inverse matrix, find the matrix X.**

9. **If A =** $\begin{pmatrix} 0 & -2 \\ 3 & 4 \end{pmatrix}$ $\boldsymbol{and\ B} =$ $\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$**, verify that** $(AB)^{-1} = B^{-1}A^{-1}$

.

10. **Find the adjoint of the matrix A =** $\begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$ **and hence show A (adj A) =** $|A|\ I_3$. **Also compute** A^{-1}

LEVEL: III

1. **State the matrix A =** $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$ **satisfies the eqn** $A^2 - 4A - 5I_3 = 0$ **and hence find** A^{-1}
2. **If the matrix A =** $\begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ 2x & -y & z \end{pmatrix}$ **satisfies the eqn** $A^T = A^{-1}$**, find the value of x , y , z .**
   (Hint : Since $A^T = A^{-1} \Rightarrow A.A^T = I = A.A^{-1}$**)**
3. **If A =** $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ **,find** A^{-1} **and Show that** $A^{-1} = \frac{1}{2}(A^2 - 3I)$
4. **If A =** $\begin{pmatrix} 0 & 1 & 3 \\ 0 & 2 & x \\ 2 & 3 & 1 \end{pmatrix}$ **and if** $A^{-1} = \begin{pmatrix} 1/2 & -4 & 5/2 \\ -1/2 & 3 & -3/2 \\ 1/2 & y & 1/2 \end{pmatrix}$.
   Find x,y .

CASE STUDY QUESTIONS

1. A manufacturer produces three stationary products Pencil, Eraser and Sharpener which he sells in two markets. Annual sales are indicated below:

Market	Products(in numbers)		
	Pencil	Eraser	Sharpener
A	10,000	2000	18,000
B	6000	20,000	8000

If the unit Sale price of Pencil, Eraser and Sharpener are Rs.2.50, Rs.1.50 and Re.1.00 respectively, and unit cost of the above three commodities are Rs.2.00, Re.1.00 and Rs.0.50 respectively, the, based on the above information, answer the following:

(i)Total revenue of market A

(a) Rs.64,000 (b) Rs.60,400 (c) Rs.46,000 (d) Rs.40,600

(ii) Total revenue of market B

(a) Rs.35,000 (b) Rs.53,000 (c) Rs.50,300 (d) Rs.30,500

(iii)Cost incurred in market A

(a) Rs.13,000 (b) Rs. 30,100 (c) Rs. 10,300 (d) Rs.31,000

(iv) Gross profit in both market

Rs.23,000 (b) Rs.20,300 (c) Rs.32,000 (d) Rs.30,200

CCT questions

Chapter 4

Determinants

Solving linear equations in three variables by matrix method

A woman of a family went to a grocery shop to purchase edible items and noticed the cost of certain items which is as follows .The cost of 4 kg onion 3kg wheat and 2kg rice is Rs 60. The cost of 2 kg onion 4kg wheat and 2kg rice is Rs 90. The cost of 6 kg onion 2kg wheat and 3 kg rice is Rs 70.

Based on the above information, answer the following questions

(1) Formulate the above situation into mathematical equations
(2) Check whether the linear equations formed above have unique solutions or not
(3) Find the adjoint of the coeffient matrix of the above equations
(4) Find the inverse of the coefficient matrix of the above equations
(5) Find the cost of each item(per kg)

THE COMMON MISTAKES COMMITTED BY THE STUDENTS IN

MATRICES AND DETERMINANTS:

1. While solving the questions based on equality of determinants students equate the corresponding elements whereas the question must be solved by equating the values of determinants.

Mistakes of signs in calculating the cofactors

For example-

Evaluate

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

Rectification-

Minors and cofactors should be explained properly before expansion of determinants

2-Mistake

students generaly using expansion method in place of using properties of determinants.

Rectification – practice should be done so that students can apply the properties of determinants.

3- Mistake

Students write the operation but the same is not implemented during solving the problem.

Evaluate

$$\begin{vmatrix} 1 & w & w2 \\ w & w2 & 1 \\ w2 & 1 & w \end{vmatrix} = 0 \quad \text{where w is the cube root of unity}$$

Applying R1 -> R1+R2+R3

$$\begin{vmatrix} 1+w+w2 & w & w2 \\ 1+w+w2 & w2 & 1 \\ 1+w+w2 & 1 & w \end{vmatrix}$$

Rectification - difference between Row and Column operation should be explained clearly and more practice is required.

4- Mistake

Mistakes found in finding the common in determinants.

For example-

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 2 & 7 \\ 8 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 & 6 \\ 5 & 1 & 7 \\ 8 & 1 & 1 \end{vmatrix}$$

Rectification –

Common should be taken either from row or column , it should not be taken from the diagonal elements.

5- Mistake

Mistakes found in breaking the determinants.

For example-

$$\begin{vmatrix} a+1 & b+1 & c+1 \\ a+2 & b+2 & c+2 \\ a+3 & b+3 & c+3 \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

Rectification –

Breaking of determinants should be explained clearly with help of different type of examples.

Let A be 3x3 matrix and IAI=5,then find I2AI.

Wrong answer:

I2AI=2x IAI=2x5=10

MISCONCEPTIONS:

Instead of applying property of order of determinants.The students multiplying both of them.

Correct Answer:

$$I2AI=2^3x\ IAI=8x5=40$$

Teaching strategy:

While telling the multiplication of constant with determinants the teacher should make it clear that if each element of a row or column of a determinant is multiplied by a scaler k, then the value of the determinant is multiplied by k. If A is a square matrix of order n, then $|kA| = k^n\ |A|$

Q. Find the cofactors of a_{12} of the determinants $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

Wrong answer:

Cofactors of a_{12} = $\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix}$ = -42 -4 =-46

MISCONCEPTIONS:

Instead of applying property of finding the cofactor of a determinant the students did not find the sign of given cofactor.

Correct Answer:

Cofactors of a_{12} = $(-1)^{1+2}$ $\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix}$ = (-1)(-42 -4) = 46

Teaching strategy:

While we explaining the concept of finding cofactor of a determinant we must clear that the difference between minor and cofactors.

CONTINUITY AND DIFFERENTIABILITY

1. Knowledge of functions :

(i) Polynomial functions: e.g. $f(x) = x^2 + 2x + 5$

(ii) Modulus function : $f(x) = |x|$

(iii) Greatest Integer Function : $f(x) = [x]$

(iv) Signum function : The signum function, denoted sgn, is defined as follows: $sgn(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$

(v) Trigonometric functions : sin x, cos x etc.

(vi) Inverse Trigonometric functions : $\sin^{-1} x$, $\cos^{-1} x$ etc.

(vii) Logarithmic functions : $f(x) = \log x$

(viii) Exponential functions : $f(x) = e^x$

2. Continuity & Discontinuity :

Continuous at a Point

The function f is **continuous at the point a in its domain** if:

1. $\lim_{x\to a} f(x)$,
2. $\lim_{x\to a} f(x) = f(a)$

If f is not continuous at a, we say that f is **discontinuous** at a.

Note

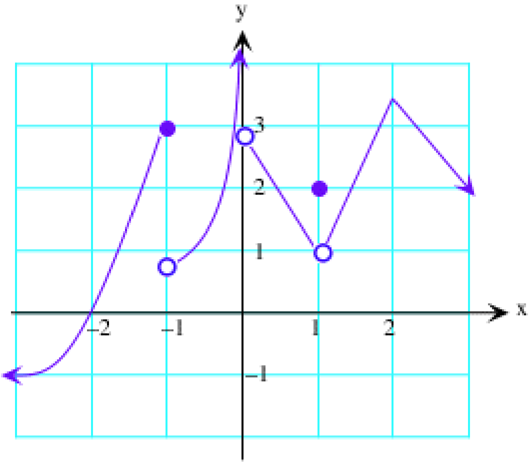
If the point a is not in the domain of f, we do not talk about whether or not f is continuous at a.

Continuous on a Subset of the Domain

The function f is **continuous on the subset S of its domain** if it continuous at each point of S.

Points of Discontinuity in a Graph

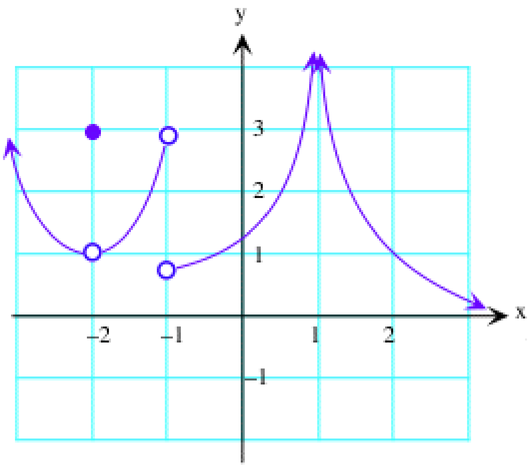
Let f have the graph shown below.



Looking at the figure, we see that the possible points of discontinuity at x = -1, 0, 1, and 2.

Points of Discontinuity in a Graph

Let f have the graph shown below.



function is discontinuous at x = - 1 and 1.

Note:

1. All polynomial functions are continuous.
2. All rational functions are continuous provided denominator does not vanish.
3. All trigonometric functions are continuous.
4. All exponential functions are continuous.

Derivative; Differentiability

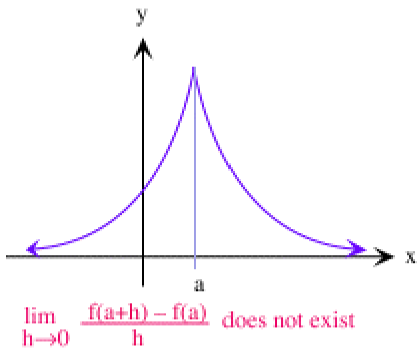
1) The **derivative** of the function f at the point a in its domain is given by

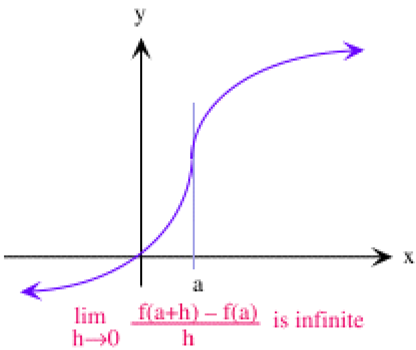
$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

2) The function f is **differentiable at the point a in its domain** if f'(a) exists.
3) The function f is **differentiable on the subset S of its domain** if it differentiable at each point of S.

4)A function can fail to be differentiable at a point a if either $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ does not exist, or is infinite

Note In the former case, we sometimes have a cusp on the graph, and in the latter case, we get a point of vertical tangency.





As we see in the above graph on the right, there are no points of vertical tangency or cusps.

As we can see, the graphs provide immediate information as to where to look for a point of non-differentiability: a point where there appears to be a cusp or a vertical tangent.
(a) Not all continuous functions are differentiable. For instance, the closed-form function f(x) = |x| is continuous at every real number (including x = 0), but not differentiable at x = 0.
(b) However, every differentiable function is continuous.

3. Derivatives of different types of functions :
   (i) Direct formulae based

   Y = sin x

   $\frac{dy}{dx}$ = cos x , etc.

   (ii) Chain Rule :

   Y = f (u)

   $$\frac{dy}{dx} = f'(u)\frac{du}{dx}$$

(iii) By taking Logarithmic

$Y = x^x$

$\log y = x \log x$

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \log x$

$\frac{dy}{dx} = y\,(1 + \log x)$

$\frac{dy}{dx} = x^x\,(1 + \log x)$

4. (a, b). Then there is at least one point c in (a, b) where $f'(c) = \frac{f(b)-f(a)}{b-a}$.

	CONTINUITY AND DIFFERENTIBILITY
	Level I
1	Determine the vaue of ' k'for which the following function is continuous at x=3. f(x) = $\left\{\frac{(x+3)^2-36}{x-3}\right\}$,x$\neq 3$, k , if x=3 (Ans k=12)
2	Determine the value of the constant 'k ' so that the function f(x) = $\left\{\frac{kx}{\lvert x\rvert}\right\}$, if x$< 0$,3 if x≥ 0 continuous at x=0. (Ans k=-3)
3	If the following function f(x) is continuous at x =0, then write the value of k. f(x) =$\left\{\frac{sin\frac{3x}{2}}{x}\right\}$,x$\neq 0$, k if x=0. (Ans k=$\frac{3}{2}$)
4	For what' k' is the function f(x) = $\left\{\frac{sin5x}{3x} + cosx\right\}$, if x$\neq 0$, $k\ if\ x = 0$ Continuous at at x=0? (Ans $\frac{8}{3}$)
5	Find all points of discontinuity of f, where f is defined by f(x)= 2x+3if x≤ 2, $2x - 3\ if\ x > 2.$ (Ans It is discontinuous at x=2).
	Find the derivative of the following functions w.r.t x (6-10)
6	Sin(3x+5)
7	e^{tanx}
8	Cos(sinx)
9	2^{x^3}
10	$\cot^{-1}(cosecx + cotx)$
11	If 2x + 5y = cos y , find $\frac{dy}{dx}$
12	If $5y^2 = 2x^3 - 5y$, find $\frac{dy}{dx}$
13	If ax + by = sin y, find $\frac{dy}{dx}$
14	If sin y + y^3 = tan x, find $\frac{dy}{dx}$
15	If y + sec y = cot x, find $\frac{dy}{dx}$

16	Differentiate with respect to x $\cos^{-1}(\sin x)$
17	Differentiate with respect to x $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$
18	Find $\frac{dy}{dx}$ if $y = \sin^{-1}(x) + \sin^{-1}\left(\sqrt{1-x^2}\right)$
19	If $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $0 < x < 1$ find $\frac{dy}{dx}$
20	If $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, $-1 < x < 1$ $find$ $\frac{dy}{dx}$
21	If $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$, $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ find $\frac{dy}{dx}$
22	Find $\frac{dy}{dx}$ if y = x^{logx}
23	Find $\frac{dy}{dx}$ if y = x^{xcosx}
24	Find $\frac{dy}{dx}$ if y = $(logx)^{cosx}$
25	If y^x = e^{y-x} P.T $\frac{dy}{dx}$ = $\frac{(1+logy)^2}{logy}$
26	If (x^y) (y^x) = 1. Find $\frac{dy}{dx}$
27	If x = at^2 and y = 2at, Find $\frac{dy}{dx}$
28	If x = 2 Cosα and y = b sinα, Find $\frac{dy}{dx}$
29	Find $\frac{dy}{dx}$, If x = a Sec$^3\theta$ and y = atan$^3\theta$
30	If x = t + (1/t), y = t – (1/t), find $\frac{dy}{dx}$,
31	Find $\frac{dy}{dx}$ If $x = a(1 - cos\vartheta), y = a(\vartheta + sin\vartheta$
32	Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.
33	If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$
34	Find the second order derivative $\log_7$(logx) with respect to x.
35	Find the second order derivative of Sin^3 x + cos^6x
36	. If x = 2at and $y = at^2$, where a is a constant then $\frac{d^2y}{dx^2}$ at $x = \frac{1}{2}$ is a. $\frac{1}{2a}$ b. 1 c. 2a d. $\frac{a}{2}$

	Level II
1	Prove that the function f(x)=5x-3 iscontinuous at x=3.
2	Find the value of 'k'for which the function f(x)=k if x=2, f(x)=$\left\{\frac{x^2+3x-10}{x-2}\right\}$, x$\neq 2$ $is\ continuous\ at\ x = 2.$ (Ans k=7)
3	Find the value of p for which the function f(x) =$\left\{\frac{1-cos4x}{x^2}\right\}$,x $\neq 0,$ = p if x=0 is continuous at x=0. (Ans p=8.)
4	Find the relationship between a and b so that the function f defined by f(x) ={ax+1if ≤ 3 $bx + 3\ if\ x > 3$ is continuous at x=3. (Ans a=b+$\frac{2}{3}$).
6	If $y = \sqrt{x^2+1}$ then $\frac{d^2y}{dx^2}$ is a) $\frac{1}{y}$ b) $\frac{1}{y^2}$ c) $\frac{1}{y^3}$ d) $\frac{1}{y^4}$
7	Find the value of $f'(x), if\ f(x) = \sin^{-1}(1-x^2)$ a)$\frac{1}{\sqrt{1-x^2}}$ b) $\frac{2}{\sqrt{2-x^2}}$ $c)\frac{-2}{\sqrt{2-x^2}}$ $d)\frac{-1}{\sqrt{1-x^2}}$
8	If y= sin(logcosx) then $\frac{dy}{dx}$ is a) cos(log(cosx))tanx b)sin(log(cosx))tanx c) -cos(log(cosx))tanx d) -cos(log(sinx))tanx
9	If $f(x) = \sqrt{tanx^2}$, then the value of $f'\left(\frac{\pi}{2}\right)\ is$ a. $\sqrt{\frac{\pi}{2}}$ b) $\sqrt{\pi}$ c) $2\sqrt{\pi}$ $d)\ 2$
10	What is the value of $\frac{d}{dx}(cos^2xtanx)at\ x = 0$ a) -1 b) 0 c) -2 d) 1
11	xy + y^2 = tan x – 2y, find $\frac{dy}{dx}$
12	If $\sin^2 x + \tan^2 y = 1$, find $\frac{dy}{dx}$
13	If $5x^3 = -3xy + 2$, find $\frac{dy}{dx}$
14	If $x^2 + xy^2 = 10$, find $\frac{dy}{dx}$
15	If $3x^2y^2 = 4x^2 - 4xy$, find $\frac{dy}{dx}$

16	Find $\frac{dy}{dx}$ if $y = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right),\ 0 < x < \frac{\pi}{2}$
17	Find $\frac{dy}{dx}$ if $y = \tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right),\ 0 < x < \frac{\pi}{2}$
18	Find $\frac{dy}{dx}$ if $y = \cot^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$
19	Find $\frac{dy}{dx}$ if $y = \tan^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$
20	Find $\frac{dy}{dx}$ if $y = \cot^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$
21	If $x = \sqrt{a^{sin^{-1}t}}$ and $y = \sqrt{a^{cos^{-1}t}}$ Show that $\frac{dy}{dx} = -\frac{y}{x}$
22	If $x^p y^q = (x + y)^{p+q}$ P.T. $\frac{dy}{dx} = \frac{y}{x}$
23	If $x^y + y^x = a^b$ find $\frac{dy}{dx}$
24	If $x^y = e^{x-y}$ Find $\frac{dy}{dx}$
25	If $x^{13} y^7 = (x + y)^{20}$ P.T $\frac{dy}{dx} = \frac{y}{x}$
26	If $(x^y)(y^x) = 1$. Find $\frac{dy}{dx}$
27	$If\ x = e^{cos2t}, y = e^{sin2t}$,prove that $\frac{dy}{dx} = -\frac{ylogx}{xlogy}$
28	If $x = a\left(cost + logtan\frac{t}{2}\right), y = asint$,find $\frac{dy}{dx}$
29	If $x = a(cost + tsint), y = a(sint - tcost)$ find $\frac{d^2y}{dx^2}$
30	$If\ x = e^{\vartheta}\left(\vartheta + \frac{1}{\vartheta}\right), y = e^{-\vartheta}\left(\vartheta - \frac{1}{\vartheta}\right)$,find $\frac{dy}{dx}$
31	If $y = \sin^{-1} x$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.
32	If $y = \cos^{-1} x$, Find $\frac{d^2y}{dx^2}$ in terms of y alone.
33	If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$
34	If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$
35	If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$.

	Level III
1	Find the value of k, so that the function f is continuous at x=$\frac{\pi}{2}$. f(x) =$\left\{\frac{kcosx}{\pi-2x}\right\}$,if x$\neq\frac{\pi}{2}$, = 5 if x=$\frac{\pi}{2}$. (Ans k=10).
2	Find all points of discontinuity of f, where f is defined as follows f(x)= $\{\lvert x\rvert+3\}$, x ≤ -3, = $-2x$, $-3 < x < 3$, = $6x+2$ $x \geq 3$. (Ans x=3).
3	Find the value of a and b such that the functions defined as follows is continuous . f(x) =x +2, x ≤ 2, = $ax+b$, $2 < x < 5$, = $3x-2$, $x \geq 5$. (Ans a=3,b=-2).
4	Find the value of a and b such that the following function f (x)=5, x≤ 2, = ax+b, 2<x < 10, = 21, $x \geq 10$., (Ans a=2,b=1)
5	Discuss the continuity of the function f(x) is defined as follows f(x) = $\frac{1}{2}$ +x , $0 \leq x < \frac{1}{2}$, =1 if x=$\frac{1}{2}$, = $\frac{3}{2}$ +x, if $\frac{1}{2} < x \leq 1$ $(Ans\ at\ x=\frac{1}{2})$
6	$If\ y = cosx + logtan\frac{x}{2}, prove\ that\ \frac{dy}{dx} = cotxcosecx$
7	If $y=\sqrt{x+1}+\sqrt{x-1}, prove\ that\ \left(\sqrt{x^2-1}\right)y_1=\frac{1}{2}y$
8	Prove that $\frac{d}{dx}\left[logtan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]=secx$
9	If y= $x\sin^{-1}x+\sqrt{1-x^2}$, prove that $\frac{dy}{dx}=\sin^{-1}x$
10	$if\ y=e^x+e^{-x}$, prove that $\frac{dy}{dx}=\sqrt{y^2-4}$
11	$If\ y=\frac{1}{2}log\left(\frac{1-cos2x}{1+cos2x}\right), prove\ that\ \frac{dy}{dx}=2cosec2x$
12	If $\sin^2 x+\tan^2 y=1$, find $\frac{dy}{dx}$
13	If $x^2+x^2y+xy^2+y^3=100$, find $\frac{dy}{dx}$
14	If $x^2+y^2=\log[\sin(xy)]$, find $\frac{dy}{dx}$

15	If $2x^3 = (3xy + 1)^2$, find $\frac{dy}{dx}$
16	$y = \cos^{-1}\left(\frac{3x+4\sqrt{1-x^2}}{5}\right)$, find $\frac{dy}{dx}$
17	$y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ find $\frac{dy}{dx}$
18	$y = \sin^{-1}\left(\frac{2.6^x}{1+36^x}\right)$, find $\frac{dy}{dx}$
19	$y = \sin^{-1}\left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right)$ find $\frac{dy}{dx}$
20	If $\tan^{-1}\frac{x^2-y^2}{x^2+y^2} = a$ then prove that $\frac{dy}{dx} = \frac{y}{x}$
21	If $y = \left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$ Find $\frac{dy}{dx}$
22	If $y = (1+x)(1+x^2)(1+x^3)(1+x^4)$ find $\frac{dy}{dx}$
23	If $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ Find $\frac{dy}{dx}$
24	If $y = \log_7 logx$. What is $\frac{dy}{dx}$
25	If $(\cos x)^y = (\sin y)^x$ then find $\frac{dy}{dx}$
26	$If\ x = \sin^{-1}\frac{2t}{1+t^2}, y = \tan^{-1}\frac{2t}{1-t^2}$, $t > 1$,prove that $\frac{dy}{dx} = -1$
27	For a positive constant 'a' find $\frac{dy}{dx}$ where $x = \left(t + \frac{1}{t}\right)^a$, $y = a^{\left(t+\frac{1}{t}\right)}$
28	$If\ x = acos\vartheta + bsin\vartheta, y = asin\vartheta - bcos\vartheta$,then prove that $y^2\frac{d^2y}{dx^2}$ - $x\frac{dy}{dx} + y = 0$
29	$If\ x = \frac{1+logt}{t^2}, y = \frac{3+2logt}{t}$,t>0 prove that $y\frac{dy}{dx} - 2x\left(\frac{dy}{dx}\right)^2 = 1$
30	$If\ x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$ then show that $\frac{dy}{dx} + \frac{x}{y} = 0$
31	If $(x-a)^2 + (y-b)^2 = c^2$ for some $c > 0$, prove that $\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant which is independent of a and b.
32	$If\ x\sqrt{1+y} + y\sqrt{1+x} = 0\ then\ Prove\ that\ (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$
33	$If\ y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x\ then\ prove\ that\ (x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = m^2y$
34	If y= $\log\left[x + \sqrt{a^2+x^2}\right], Show\ that\ (a^2+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$
35	If $x^m y^n = (x+y)^{m+n}$ Prove that $\frac{d^2y}{dx^2} = 0$

CREATIVE AND CRITICAL THINKING (CCT)

Rama was preparing for her upcoming board exam. Her elder sister was helping her in doing so. He asked her the following questions.

i) If the function $f(x) = \begin{cases} \frac{sin10x}{x}, x \neq 0 \end{cases}$ is continuous at x = 0 then what is the value of $f(0)$?

ii) Is every continuous function differentiable?

iii) Is every differentiable function continuous?

iv) Define differentiability of a function at a point.

A potter made a mud vessel, where the shape of the pot is based on f(x) = |x – 3| + | x-2|, where f(x) represents the height of the pot.

1. When x > 4 What will be the height in terms of x ?

a. x – 2

b. x -3

c. 2x -5

d. 5 – 2x

2. Will the slope vary with x value?

a. Yes

b. No

3. What is $\frac{dy}{dx}$ at x = 3

a. 2

b. -2

c. Function is not differentiable

d. 1

4. When the x value lies between (2,3) then the function is

a. 2x -5

b. 5 – 2x

c. 1

d.5

5. If the potter is trying to make a pot using the function $f(x) = [x]$,will he get a pot or not? Why?

a. Yes, because it is a continuous function

b. Yes, because it is not continuous

c. No , because it is a continuous function

d. No , because it is not continuous

COMMON ERRORS COMMITTED BY STUDENTS IN

CONTINUITY & DIFFERENTIABILITY

Q.1.Given a function $\mathrm{f(x)} = \frac{x^2-9}{3x-9}$, answer the following

(a) What happen to the graph of f at x=3

(b) What is the limit of the function f at x = 3

(c) What is the value of f at x = 3

(d) Is the function (cont./ discont.) at x = 3.

Student wrong answer: (a) May answered without sketching the graph that it represents a straight line.

(b) $\mathrm{f(x)} = \frac{x^2-9}{3x-9} = \frac{(x-3)(x+3)}{3(x-3)} = \frac{x+3}{3}$

$\lim_{x\to 3} f(x) = \lim_{x\to 3+} \frac{x+3}{3}$ =2

$\lim_{x\to 3-} \frac{x+3}{3} = \frac{-3+3}{3} = 0$

Hence limit does not exist

(c) $\frac{0}{0} = 0$

(d) Discontinuous

Correct Answer: (a) Break in graph at x =3

(b) $\lim_{x\to} \frac{(x-3)(x+3)}{3(x-3)} = \lim_{x\to 3} \frac{x+3}{3} = 2$

(c) Not defined

(d) Discontinuous

Q.2 Given a function f such that $\lim_{x\to 2} f(x) = 3$,which of the following statements must be true about the function f.

(i) f(2) = 3 (ii) f is continuous at x = 2 (iii) f(x) is defined at x= 2 (iv) $\lim_{x\to 2+} f(x) = 3$ (v) None

Student wrong answer: (ii)

Correct answer: (iv)

Q.3 Show that the function defined by f(x) =$\sin x^2$ is a continuous function.

Student answer: $\lim_{x\to a} \sin x^2 = \lim_{x\to a-} \sin x^2 = \lim_{h\to 0} \sin(a-h)^2$ =$-\sin a^2$

$\lim_{x\to a+} \sin x^2$ =$\lim_{h\to 0} \sin(a+h)^2$=$\sin a^2$

and f(a) = $\sin a^2$

Correct answer: let g(x) = $\sin x$, h(x) = x^2 , goh(x) = g{h(x)} = g(x^2) = $\sin x^2$

The function f is considered as the composition goh of two functions g and f. Since g and f are both continuous functions. Hence f is continuous function.

Q.4 Examine that $\sin|x|$ is a continuous function.

May be explained in similar way as above.

Teaching strategy: To start the topic with two examples to feel of continuity.

(i) $\mathrm{f(x)} = \begin{cases} 1, & x \leq 0 \\ 2, & x > 0 \end{cases}$ (ii) $\mathrm{f(x)} = \begin{cases} 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$

then mathematically continuity may be defined as the function f is continuous at x =a if $\lim_{x\to a} f(x) = f(a)$ or $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = f(a)$

Find the derivative of $y = x^x$ w.r.t. x

Wrong Answer:

$$y' = x.x^{x-1}$$

Misconception: Instead of applying Logarithmic Differentiation, the student has applied the formula for derivative of x^n without realizing that in this formula *n* is a constant and not a variable.

Correct Answer:

$$y = x^x$$

$$\Rightarrow \log y = x \log x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + 1.\log x$$

$$\Rightarrow \frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x)$$

Teaching Strategy: While telling the formula for derivative of x^n, the teacher should make it clear that here *n* is a constant and not a variable and while teaching logarithmic differentiation, it should be made clear that whenever variable *x* is in the power, the student should apply logarithms on both sides and then proceed for logarithmic differentiation.

Differentiate the function $xy + x^2 = \sin y^2$ w.r.t. *x*

Wrong Answer:

$$1.y + x.\frac{dy}{dx} + 2x = \cos y^2$$

Misconception: Chain Rule has not been applied while solving the problem. The student has merely done derivative of sin as cos, but he did not apply chain rule to calculate the derivative of y^2.

Correct Answer:

$$1.y + x.\frac{dy}{dx} + 2x = \cos y^2 . 2y.\frac{dy}{dx}$$
$$\Rightarrow (2y \cos y^2 - x)\frac{dy}{dx} = y + 2x$$
$$\Rightarrow \frac{dy}{dx} = \frac{y + 2x}{2y \cos y^2 - x}$$

Teaching Strategy: It should be made clear to the students that while applying differentiation to composite functions; chain rule is to be applied with utmost care. While applying chain rule it must be noted that first the derivative of outer most function is to be done followed by inner functions.

Differentiate the function $xy + x^2 = \sin y^2$ w.r.t. x

Wrong Answer:

$$1.y + x.\frac{dy}{dx} + 2x = 2\sin y\ \frac{dy}{dx}$$

Misconception: This time though the student has applied chain rule but he has wrongly interpreted $\sin y^2$ as $(\sin y)^2$ instead of $\sin(y^2)$.

Correct Answer:

$$1.y + x.\frac{dy}{dx} + 2x = \cos y^2\,.2y.\frac{dy}{dx}$$
$$\Rightarrow (2y\cos y^2 - x)\frac{dy}{dx} = y + 2x$$
$$\Rightarrow \frac{dy}{dx} = \frac{y + 2x}{2y\cos y^2 - x}$$

Teaching Strategy: It should be made clear to the students in class XI itself, while teaching Trigonometric Functions, that there is a difference between $(\sin y)^2$ and $\sin(y^2)$. In the first expression first sine of *y* is applied after which the result is squared while in the second expression first square of the angle is done and afterwards, sine is applied on the result. In this question it is $\sin y^2$ which is equal to $\sin(y^2)$ and so first the derivative of sine and afterwards the derivative of y^2 is to be applied.

Find $\frac{d^2y}{dx^2}$ if $x = t^2$ and $y = t^3$.

Wrong Answer:

$$\frac{dx}{dt} = 2t \ \text{ and } \ \frac{dy}{dt} = 3t^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3t}{2}$$
$$\Rightarrow \frac{d^2y}{dx^2} = 0$$

Misconception: While calculating the second derivative, *t* has been treated as a constant and not as a parameter.

Correct Answer:

$$\frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 3t^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}} = \frac{3t^2}{2t} = \frac{3t}{2}$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{3t}{2}\right) = \frac{3}{2}\frac{dt}{dx} = \frac{3}{2}\left(\frac{1}{dt/_{dx}}\right) = \frac{3}{2}.\frac{1}{2t} = \frac{3}{4t}$$

Teaching Strategy: The teacher should explain that t is a parameter and not a constant and therefore while calculating higher order derivatives care should be taken to treat t as a function of x and its derivative must be taken into consideration.

Find $\frac{dy}{dx}$ if $y = x^x + \sin x$

Wrong Answer:

$$y = x^x + \sin x$$
$$\Rightarrow \log y = \log(x^x + \sin x) = \log x^x + \log\ \sin x$$
$$\Rightarrow \log y = x \log x + \log\ \sin x$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + 1.\log x + \frac{\cos x}{\sin x}$$
$$\Rightarrow \frac{dy}{dx} = y(1 + \log x + \cot x)$$
$$\Rightarrow \frac{dy}{dx} = (x^x + \sin x)(1 + \log x + \cot x)$$

Misconception: The student has wrong notion of logarithmic identities. He has wrongly taken $\log(m + n) = \log m + \log n$.

Correct Answer:

$$y = x^x + \sin x$$
$$\text{Let } u = x^x \text{ and } v = \sin x$$
$$\Rightarrow y = u + v$$
$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$
$$\text{Now, } \log u = x \log x \text{ and } v = \sin x$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x) \text{ and } \frac{dv}{dx} = \cos x$$

$$\Rightarrow \frac{dy}{dx} = x^x(1 + \log x) + \cos x$$

Teaching Strategy: The teacher should make it very clear that logarithmic identities must be applied with care and that $\log(m + n) \neq \log m + \log n$.

Instead log *mn* = log *m* + log *n*

S. No.	Errors	Correction	Remarks
1.	$y = x^y + y^x$ $logy = logx^y + logy^x$	$y = x^y + y^x$ $y = u + v$ $u = x^y, log\, u = y\, log\, x$ & $v = y^x, log\, v = x\, log\, y$	Proper application of logarithmic properties
2.	$\frac{d(log_e e)}{dx} = \frac{1}{e}$	$log_e e = 1$ $\frac{d(1)}{dx} = 0$	Proper application of logarithmic properties
3.	$\frac{d}{dx}(x^x) = x^x$	$\frac{d}{dx}(x^x) = x^x(1 + logx)$	Proper application of logarithmic properties
4.	$\frac{d}{dx}(x^x) = x.x^{x-1}$	$\frac{d}{dx}(x^x) = x^x(1 + logx)$	Proper application of logarithmic properties
5.	$\frac{d(\sin 2x)}{dx} = \cos 2x$	$\frac{d(\sin 2x)}{dx} = 2\cos 2x$	Application of chain rule
6.	$\frac{d}{dx}(a^x) = x.a^{x-1}$	$\frac{d}{dx}(a^x) = a^x log_e a$	Proper application of logarithmic properties
7.	$\frac{d}{dx}(a^x) = a^x$	$\frac{d}{dx}(a^x) = a^x log_e a$	Proper application of logarithmic properties
8.	$\frac{dy}{dx}$ **= sin t** $\frac{d^2y}{dx^2} = \cos t$	$\frac{dy}{dx}$ **= sin t** $\frac{d^2y}{dx^2} = \cos t.\frac{dt}{dx}$	Application of chain rule
9.	$If\ x = f(t) and\ y = g(t),$ $\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$	$\frac{d^2y}{dx^2} = \frac{d(f(t))}{dt}.\frac{dt}{dx}$	Application of chain rule
10.	Differentiate $f(x) = (x-1)^{\frac{2}{3}}$ on [0,2]. Answer: $f'(x) = \frac{2}{3}(x-1)^{\frac{-1}{3}}$	f′(x) $= \frac{2}{3}(x-1)^{\frac{-1}{3}}$, but Left hand derivative and right hand derivative at x = 1 are not equal. So at x = 1 it is not derivable and hence not differentiable in (0, 2)	Check LHD and RHD.
11.	$f(x) = \frac{1}{x}$ then $f'(x) = \frac{1\frac{d}{dx}x - x\frac{d}{dx}1}{x^2}$	$f'(x) = \frac{x\frac{d}{dx}1 - 1\frac{d}{dx}x}{x^2} = \frac{1}{x^2}$ $or\ f'(x) = \frac{d(x^{-1})}{dx} = -1\, x^{-2}$ $= \frac{-1}{x^2}$	Apply correct way of quotient rule.

Topic	question	wrong answer	Type of error	Correct answer	Follow up
Chain Rule	Differentiate sin(3x + 5) w.r.t. x	Cos (3x + 5)	conceptual	$\frac{d}{dx}$cos(3x+5)× $\frac{d}{dx}$(3x + 5)	$Find\ \frac{dy}{dx}\ if$ i) $y = \sqrt{sinx}$ ii) $y = e^{2x+3}$ iii) $y = \log(3x+5)$ iv) y=cosec(2x-5) v) $y = (5x+4)^4$
Second order derivatives	If x= a(cost+ tsint) , y= a(sint tcost) find $\frac{d^2y}{dx^2}$	$\frac{d^2y}{dx^2} = sec^2t$	conceptual	$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$ $= sec^2t \times \frac{1}{atcost}$ $= \frac{1}{atcos^3t}$	Find $\frac{d^2y}{dx^2}$ $i)\ x = 2at, y = at^2$ ii) $x = t, y = \frac{1}{t}$ iii) $x = cost$, $y = sint$ iv) x= atant , y=asect
Logarithmic differentiation	Find $\frac{dy}{dx}$ if $x^y + y^x$=a^b	logx^y + $logy^x = loga^b$	Conceptional procedural	$x^y + y^x$=a^b u+v=a^b differentiating w.r.t.x $\frac{du}{dx} + \frac{dv}{dx} = 0$ Where u=x^y,v=y^x Then logarithmic differentiation	Find $\frac{dy}{dx}$ if $x^y + x^x$=a^b $x^y - x^x$=a^b
Implicit Functions	Find $\frac{dy}{dx}$ if $x^2 +$ $xy + y^2$=9	Differentiating w.r.t x 2x+ $\frac{dy}{dx}$ +2y=0	Conceptional procedural	2 x + $x\frac{dy}{dx}$+ y +2 y $\frac{dy}{dx}$ =0	Find $\frac{dy}{dx}$ if $i)x^2 + y^2$=1 $ii)x^3 + y^3$=1

APPLICATION OF DERIVATIVES

1. RATE OF CHANGE OF QUANTITY:

1) $\frac{dy}{dx}$ represents the rate of change of y with respect to x.
   Here y is the depending variable and x is the independent variable.
2) $\frac{dy}{dx}$ at a particular point x_0 represents the rate of change of y w.r.t. x at x = x_0.
3) If x = f(t) and y = g(t), then $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$.
4) Marginal cost represents the instantaneous rate of change of the total cost at any level of output. If C(x) represents the cost function for x units produced, then marginal cost, denoted by MC, is given by MC = $\frac{d}{dx}\{C(x)\}$.
5) Marginal revenue represents the rate of change of total re4venue with respect to the number of items sold at an instant. If R(x) is the revenue function for x units sold , then marginal revenue , denoted by MR, is given by MR = $\frac{d}{dx}\{R(x)\}$.
6) The ***profit function*** $P(x)$ is given by the formula: $P(x) = R(x) - C(x)$.
7) The rate of change of the profit function $P(x)$ is $P'(x)$ and is called the ***marginal profit functio*n** for the product.
8) The point at which the profit is zero is called the ***break-even point***.

Increasing / Decreasing Functions

1) Let $x \,\&\, x'$ be any two points taken from an interval (a,b).

2) A real function $y = f(x)$ is ***increasing*** on the (a, b), if $f(x) < f(x')$ whenever $x < x'$.

3) The function is said to be ***decreasing*** on (a,b), if $f(x) > f(x')$, whenever $x < x'$.

Increasing / decreasing test:

The following result, called *Increasing/decreasing test,* is very useful in applications:

(i) If $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on (a,b)

(ii) If $f'(x) < 0$ for all $x \in (a,b)$, then f is decreasing on (a,b)

(iii) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on (a,b).

*****:** A function f is said to be ***monotonic*** on an interval, if it is either increasing or decreasing on that interval.

5) A function $y = f(x)$ is said to have a ***critical point*** at $x = c$, if any one of the following conditions is satisfied:

(a) $f'(c) = 0$ (b) $f'(c)$ is undefined, but $f(x)$ is continuous at $x = c$.

6) Let $y = f(x)$ be a given function. The points where $f'(x) = 0$ are called ***stationary points*** of the function. So, we can find the stationary points of a function $y = f(x)$ by solving the equation $f'(x) = 0$ for x.

Maxima & Minima

1)There are two types of extreme positions: ***local (relative)*** and ***global (absolute).***

2)A function $f(x)$ defined on an interval $[a,b]$ is said to have a ***local (or relative) maximum*** at a point $x = c$, if $f(c) \geq f(c+h)$ for all sufficiently small negative as well as positive values of h. The function is said to have a ***local (or relative) minimum*** at $x = c$, if $f(c) \leq f(c+h)$.

3)The point $x = c$, where $f'(c) = 0$ or $f'(c)$ does not exist, is called a ***critical point*** of the function $f(x)$.

4)A maximum or a minimum value of a function is also termed as ***extremum*** or ***extreme value*** of the function.

5) Let f be function defined in the closed interval I. If there exist a point 'a' in the interval I such that $f(a) \geq f(x)$ for every x ϵ I, then the function is said to attain absolute maximum at x = a, and $f(a)$ is absolute maximum value.

6) Let f be function defined in the closed interval I. If there exist a point 'a' in the interval I such that $f(a) \leq f(x)$ for every x ϵ I, then the function is said to attain absolute minimum at x = a, and $f(a)$ is absolute minimum value.

7) To find the absolute maxima or minima in [a, b] we have to find out the value at the end point of interval [a, b] i.e. $f(a)$ and $f(b)$ along with local maxim or minima.

Test for maximum or minimum:

1) First Derivative Test : If a function $f(x)$ has either local maxima or minima at a point $x=c$, then either $f'(c)=0$ or $f'(c)$ does not exist, i.e, $x=c$ is a critical point of the function. Of course, there may be functions for which $f(c)$ is not a local extremum, even when $x=c$ is a critical point of the function.

2) If f(x) does not change its sign in the neighborhood of 'x_1', then 'x_1' is neither point of local maxima nor local minima, then x_1 is called the point of inflexion.

Second Derivative Test: Let $f'(c)=0$ for a given function $f(x)$ defined on (a,b). Then

(i) $f''(c)<0 \Rightarrow f(c)$ is a local maximum of $f(x)$

(ii) $f''(c)>0 \Rightarrow f(c)$ is a local minimum of $f(x)$.

NOTE: If second derivative test fails, then apply first derivative test.(In case of linear polynomial second derivative test fails, as the first derivative becomes constant and hence it's second derivative become zero.

MATHEMATICAL MODELLING OF THE WORD PROBLEM
(MAXIMA AND MINIMA)

1) Draw the labeled figure wherever it is required.
2) Write the maximizing or minimizing function (correct formula).
3) Convert the function in a single independent variable using either by previous knowledge or given data.
4) Differentiate the function got in step (3) w.r.t. that independent variable.
5) Find the critical points by equating first derivative to zero.
6) Find the second derivative got in step (3).
7) Find the value of second derivative at critical points.
8) If the value is less than zero then the function has maxima and the maximum is the value of the function(in step 3) at the critical point.
9) If the value is greater than zero then the function has minima and the minimum is the value of the function(in step 3) at the critical point.

GRADED LEVEL OF QUESTIONS
CHAPTER-6: APPLICATION OF DERIVATIVES

Q.No	LEVEL I
1	The radius of a circle is increasing uniformly at the rate of 3 cm/sec. Find the rate at which the area of the circle is increasing when the radius is 10cm.
2	The total revenue in Rs. received from the sale of x units of an article is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue when x = 15.
3	The length *x* of a rectangle is decreasing at the rate of 3 cm/minute and the width *y* is increasing at the rate of 2cm/minute. When *x* =10cm and *y* = 6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.
4	A particle moves along the curve 6*y* = *x*3 +2. Find the points on the curve at which the *y*-coordinate is changing 8 times as fast as the *x*-coordinate
5	A balloon, which always remains spherical, has a variable diameter 3/2(2x+1) Find the rate of change of its volume with respect to *x*.
6	Prove that the function $f(x) = e^x$ is strictly increasing on R.
7	Prove that the logarithmic function is strictly increasing on $(0, \infty)$
8	Prove that the function $f(x) = \log \sin x$ is strictly increasing on $(0, \frac{\pi}{2})$ and strictly decreasing on $(\frac{\pi}{2}, \pi)$.
9	Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on $(-1, 1)$.
10	Let I be an interval disjoint from $(-1, 1)$. Prove that $f(x) = x + \frac{1}{x}$ is increasing on I.
11	Find the points of local maxima and local minima of $f(x) = x\,\lvert x\rvert$
12	Find the points of local maxima and local minima of $f(x) = 3 + \lvert x\rvert, x \in R$
13	What is the local maximum value of the function $\frac{\log x}{x}$?
14	Find the local maxima and minima for the given function and also find the local maximum and local minimum values $f(x) = x^3, x \in (-2, 2)$
15	Discuss local maxima and minima of $f(x) = x^x, x > 0$

16	A metal box with a square base and vertical sides is to contain 1024 cm^3.The material for the top and bottom costs Rs 5 per $cm^2$and the material for the sides costs Rs 2.50 per cm^2..Find the least cost.
17	Find the dimensions of the rectangle of perimeter 36cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.
18	If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?
19	The sum of the surface areas of a rectangular parallelopiped with sides x,2xand x/3 and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of the sphere. Also find the minimum value of the sum of their volumes.
20	A open box is to be made out of a piece of a square card board of sides 18 cm by cutting of equal squares from the corners and turning up the sides. Find the maximum volume of the box.
21	Find the minimum value of the function $f(x) = x + \frac{1}{x}, \quad x > 0$.
22	Find the maximum and minimum value of the functions: $f(x) = -\|x+1\| + 5$.
23	Find the maximum and minimum value of the functions: $g(x) = 2 \sin x + 3$.
24	Find the maximum and minimum value of the functions: $h(x) = x + 1,\ x \in (-1, 1)$
25	It is given that $x = 1$, the function $f(x) = x^3 - 12x^2 + kx + 7$ attains maximum value in the interval [0, 2]. Find the value of k.
Q.No	**LEVEL II**
1	The surface area of a spherical bubble is increasing at the rate of 2 cm^2/sec. Find the rate at which the volume of the bubble is increasing at the instant when its radius is 6 cm .
2	Sand is pouring from a pipe at the rate of 12cm ^{3} /sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when h=4cm.
3	A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadowincreases.
4	A car starts from a point P at time t = 0 seconds and stops at point Q. The distance x, in metres, covered by it, in t seconds is given by $x = t^2\left(2 - \frac{t}{3}\right)$ $\mathit{Find\ the\ time\ taken\ by\ it\ to\ reach\ Q\ and\ also\ find\ the\ distance\ between}$

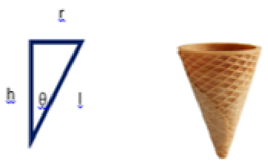
5	A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/s. Find the rate at which its area is increasing when radius is 3.2 cm
6	Find the intervals in which the function f (x) = 10 – 6 x – 2 x^2 is strictly increasing or strictly decreasing.
7	Find the intervals in which the function given by f (x) = 2 x^3 – 9 x^2 + 12x + 15 is (a) strictly increasing (b) strictly decreasing.
8	Find the intervals in which the function given by f (x) = (x + 1 $)^3$(x – 3 $)^3$ is (a) strictly increasing (b) strictly decreasing.
9	Find the value(s) of x for which y = $[x(x-2)]^2$ is an increasing function.
10	Find the intervals in which the function given by f (x) = x^2 e^{-x} is (a) strictly increasing (b) strictly decreasing.
11	Find the points of local maxima and local minima, if any, of the following function. Also find the local maximum and local minimum values,
12	$f(x) = sin2x - x$, where -$\frac{\pi}{2} < x < \frac{\pi}{2}$
13	Find the local maxima or local minima, if any, of the function $f(x) = sinx + cosx, 0 < x < \frac{\pi}{2}$ using the first derivative test.
14	Find all the points of local maxima and local minima as well as the corresponding local maximum and local minimum values for the function $f(x) = (x-1)^3(x+1)^2$.
15	Show that a local minimum value of $f(x) = x + \frac{1}{x}, x \neq 0$ is greater than a local maximum value.
16	Show that the right circular cone of least curved surface area and given volume is$\sqrt{2}$ times the radius of the base.
17	Show that the right circular cylinder of given surface area and maximum volume is such that its height is equal to the diameter of the base.
18	Of all the closed cylindrical cans of a given volumes of 100 cubic centimeters, find the dimension of the can which has the minimum surface area.
19	Show that the semi vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.
20	Show that the semi vertical angle of right circular cone of the given surface area and maximum volume is $\sin^{-1}\frac{1}{3}$
21	Find the maximum profit that a company can make if the profit function is given by $p(x) = 41 + 24x - 18x^2$.

22	What is the maximum value of the function $sin\,x + cos\,x$ in $(0, \pi)$?
23	Find two numbers whose sum is 16 and the sum of their squares is least.
24	If $y = a\,logx + bx^2 + x$ has its extreme values at $x = -1$ and $x = 2$ then find a and b.
25	Find two numbers whose sum is 24 and whose product is as large as possible.
Q.No	**LEVEL III**
1	A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is tan–1(0.5). Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m
2	The two equal sides of an isosceles triangle with fixed base *b* are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?
3	A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of (A) 1 m/h (B) 0.1 m/h (C) 1.1 m/h (D) 0.5 m/h
4	A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?
5	A kite is moving horizontally at the height of 151.5m. If the speed of the kite is 10m/sec, How fast is the string being let out , when the kite is 250m away from the boy who is flying the kite? The height of the boy is 1.5m.
6	Find the intervals in which the function given by $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is (a) strictly increasing (b) strictly decreasing.
7	Find the intervals in which the function $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is (a) strictly increasing (b) strictly decreasing.
8	Find the intervals in which the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is (a) strictly increasing (b) strictly decreasing.
9	Prove that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function in $(0, \frac{\pi}{2})$.

10	Find the intervals in which the function given by f (x) = sinx − cos x , 0 ≤ x ≤ 2 π is (a) strictly increasing (b) strictly decreasing.
11	. Find the local maximum and local minimum values of $f(x) = secx + logcos^2x, 0 < x < 2\pi$
12	Find all the points of local maxima and local minima and the corresponding maximum and minimum values of the function $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$
13	Find the local maxima and local minima for the given function and also find the local maximum and local minimum values $f(x) = x^3 - 6x^2 + 9x + 15$
14	If $f(x) = x^5 - 5x^4 + 5x^3 - 10$ has local maxima and minima at x=k and x=m respectively, then 2k+m is
15	A cone is circumscribed about a sphere of radius r. Show that the volume of the cone is maximum when its semi vertical angle is $\sin^{-1}\frac{1}{3}$
16	Show that the height of a cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$
17	Show that the volume of greatest cylinder which can be inscribed in a cone of height h and semi vertical angle α I $\frac{4}{27}\pi h^2 tan^2\alpha$
18	Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.
19	An open box with, a square base is to be made out of a given quantity of metal sheet of area c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$.
20	A point on the hypotenuse of a right triangle is at a distances of a and b from the sides. Show that the minimum length of the hypotenuse is $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.
21	A manufacturer can sell x items at price of Rs. $(250 - x)$ each. The cost of producing x items is Rs. $(2x^2 - 50x + 12)$. Determine the number of items to be sold so that he can make maximum profit.
22	A jet enemy plane is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). What is the nearest distance between the soldier and the jet ?
23	Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible when revolved about one of its sides. Also find the maximum volume.

24

36)An ice cream manufacturing company is planning to make right circular conical cups of least inner curved surface area to contain 50cm^3 of ice cream. If r, h ,l ,c and θ are respectively the radius, height , slant height ,curved surface area and semi vertical angle of the conical shaped ice cream inside the conical cup.



I. Relation between h and r is h=

(a) $\frac{150}{r^2}$ (b) $\frac{150}{\pi r^2}$ (c) $\frac{50}{\pi r^2}$ (d) none of these

II. The relation between c and r

(a) $c^2 = \frac{4500}{r^2} + \pi^2 r^4$ (b) $c^2 = \frac{22500}{r^2} + \pi^2 r^4$ (c) $c^2 = \frac{2500}{r^2} + \pi^2 r^4$ (d) none of these

III. If c^2 = D then $\frac{\mathrm{dD}}{\mathrm{dr}} =$

(a) $-\frac{5000}{r^3} + 4\pi^2 r^3$ (b) $-\frac{45000}{r^3} + 4\pi^2 r^3$ (c) $c^2 = \frac{-5000}{r^2} + 4\pi^2 r^3$ (d) none of these

IV. c^2 is minimum then the relation between h and r is

(a) h = r (b) h = 2r (c) h =$\sqrt{2}$r (d) none of the above

V. If the curved surface area of conical cup is least then semi vertical θ =

(a) 45^0 (b) 30^0 (c) 60^0 (d) none of these

TOPIC: LOCAL MAXIMA & LOCAL MINIMA PROBLEMS

LEVEL – I

1. Find the points of local maxima and local minima of $f(x) = x\,|x|$

2. Find the points of local maxima and local minima of $f(x) = 3 + |x|, x \in R$

3. What is the local maximum value of the function $\frac{logx}{x}$?

4. Find the local maxima and minima for the given function and also find the local maximum and local minimum values $f(x) = x^3, x \in (-2, 2)$

5. Discuss local maxima and minima of $f(x) = x^x, x > 0$

LEVEL – II

1. Find the points of local maxima and local minima, if any, of the following function. Also find the local maximum and local minimum values,

$$f(x) = sin2x - x, \text{ where } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

2. Find the local maxima or local minima, if any, of the function $f(x) = sinx + cosx, 0 < x < \frac{\pi}{2}$ using the first derivative test.

3. Find all the points of local maxima and local minima as well as the corresponding local maximum and local minimum values for the function $f(x) = (x-1)^3(x+1)^2$.

4. Show that a local minimum value of $f(x) = x + \frac{1}{x}, x \neq 0$ is greater than a local maximum value.

5. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima.

LEVEL-III

1. Find the local maximum and local minimum values of

$$f(x) = secx + logcos^2x, 0 < x < 2\pi$$

2. Find all the points of local maxima and local minima and the corresponding maximum and minimum values of the function $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$.

3. Find the local maxima and local minima for the given function and also find the local maximum and local minimum values $f(x) = x^3 - 6x^2 + 9x + 15$

4. if $f(x) = x^5 - 5x^4 + 5x^3 - 10$ has local maxima and minima at x=k and x=m respectively, then 2k+m is

5. Find the local maximum or local minimum, if any, of the function

$f(x) = sin^4\mathrm{x} + \cos^4\mathrm{x}, 0 < \mathrm{x} < \frac{\pi}{2}$ using the first derivative test

TOPIC: MAXIMA AND MINIMA

CCT QUESTION

In a residential society comprising of 100 houses, there were 60 children between the ages of 10 – 15 years. They were inspired by their teachers to start composting to ensure that biodegradable waste is recycled. For this purpose, instead of each child doing it for only his/her house, children convinced the Residents Welfare Association to do it as a society initiative. For this they identified a square area in the local park. Local authorities charged amount of Rs 50 per square metre for space so that there is no misuse of the space and the Residents Welfare Association takes it seriously. Association hired a labourer for digging out 250 m^3 and he charged Rs 400 x $(\text{depth})^2$. Association will like to have minimum cost.

Read on the above information, answer the following.

1. Let the side of a square plot be x m and its depth be h m, then cost c for the pit is :

   (a) $\frac{50}{h}$ + 400 h^2 (b) $\frac{12500}{h}$ + 400 h^2 (c) $\frac{250}{h}$ + h^2 (d) $\frac{250}{h}$ + 400 h^2

2. Value of h (in metre) for which $\frac{dc}{dh}$ = 0 is :

(a) 1.5 (b) 2 (c) 2.5 (d) 3

3. $\frac{d^2 c}{dh^2}$ is given by

(a) $\frac{25000}{h^3}$ + 800 (b) $\frac{500}{h^3}$ + 800 (c) $\frac{100}{h^3}$ + 800 (d) $\frac{500}{h^3}$ + 2

4 . Value of x (in metre) for minimum cost is :

(a) 5 (b) $10\sqrt{\frac{5}{3}}$ (c) $5\sqrt{5}$ (d) 10

5. Total minimum cost of digging the pit (in Rs) is :

(a) 4100 (b) 7500 (c) 7850 (d) 3220

<u>ANSWERS:</u>

1. (b) 2.(c) 3.(a) 4.(d) 5.(b)

ERROR ANAYSIS

APPLICATION OF DERIVATIVE

TOPIC	QUESTION
Increasing/Decreasing Functions	Find the intervals in which the function $f(x) =- 2x^3 +9x^2 -12x - 15$ is increasing or decreasing

SOLUTION GIVEN BY STUDENTS

$f(x)=– (2x^3 -9x^2 +12x + 15)$

$f'(x) = 0$ gives us $6(x - 1)(x - 2) = 0 \Rightarrow x = 1, 2$

The points $x = 1, 2$ divide the real line into three intervals $(-\infty, 1), (1, 2), (2, \infty)$

1. In the interval $(-\infty, 1)$, $f'(x) > 0$

$\therefore$ $f(x)$ is increasing in $(-\infty, 1)$

2. In the interval $(1, 2)$, $f'(x) < 0$

$\therefore$ $f(x)$ is decreasing in $(1, 2)$

3. In the interval $(2, \infty)$, $f'(x) > 0$

$\therefore$ $f(x)$ is increasing in $(2, \infty)$

Error

Not correctly observing the sign of f‘(x) in the interval before deciding the nature of the function

Correct answer

$f'(x) = -6(x-1)(x-2)$ in the interval $(-\infty,1)$, $f'(x)<0$ f(x) is decreasing
In the interval (1,2) f(x) is increasing In the interval $(2,\infty)$.

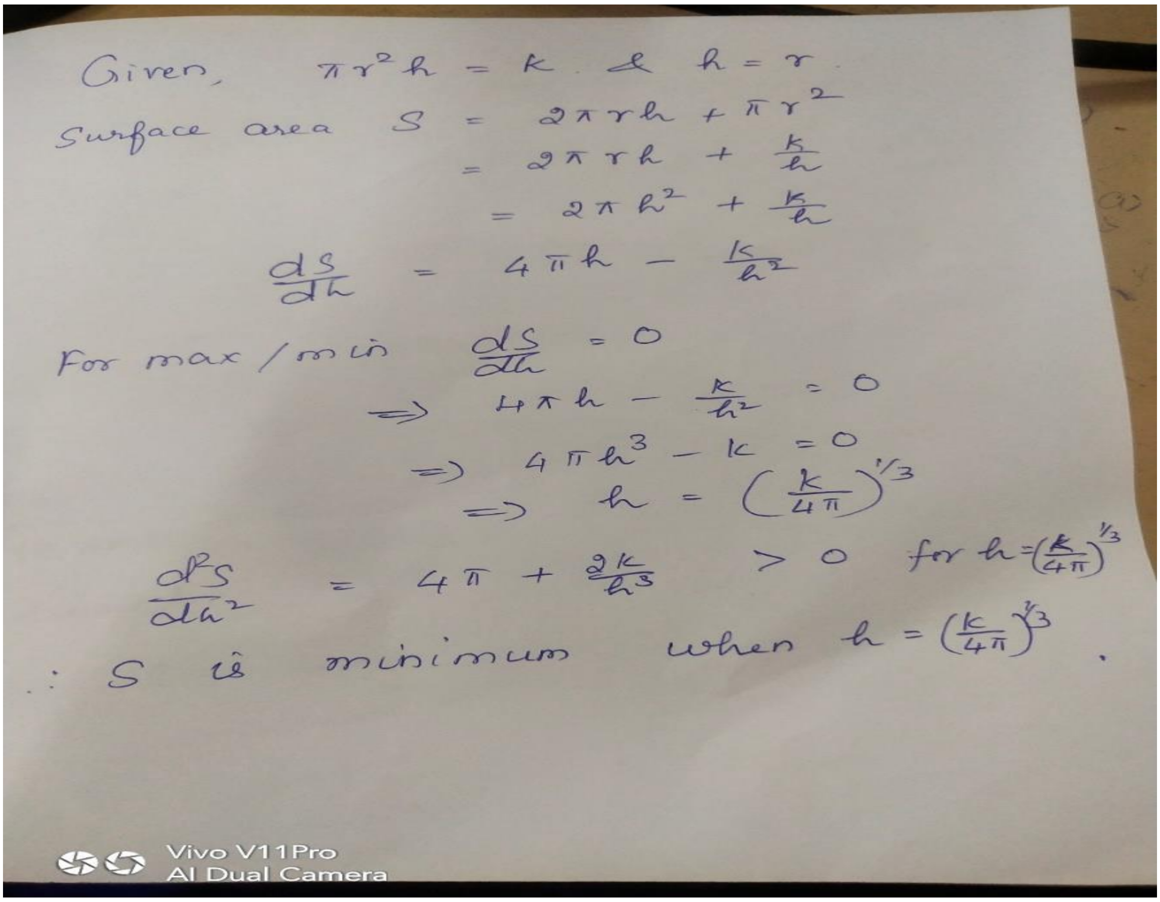
Extra problems

1. Find the intervals in which the function $f(x) =- 3x^4 - 4x^3 + 12x^2 - 12.$ is increasing or decreasing

2. Find the intervals in which the function $f(x) =- - x^3 + 6x^2 - 9x - 15.$ is increasing or decreasing

TOPIC	QUESTION
Maxima and Minima	Show that a cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base.

<u>Solution done by the students</u>



Given, $\pi r^2 h = k$ & $h = r$

Surface area $S = 2\pi r h + \pi r^2$

$= 2\pi r h + \frac{k}{h}$

$= 2\pi h^2 + \frac{k}{h}$

$\frac{dS}{dh} = 4\pi h - \frac{k}{h^2}$

For max / min $\frac{dS}{dh} = 0$

$\Rightarrow 4\pi h - \frac{k}{h^2} = 0$

$\Rightarrow 4\pi h^3 - k = 0$

$\Rightarrow h = \left(\frac{k}{4\pi}\right)^{1/3}$

$\frac{d^2S}{dh^2} = 4\pi + \frac{2k}{h^3} > 0$ for $h = \left(\frac{k}{4\pi}\right)^{1/3}$

$\therefore S$ is minimum when $h = \left(\frac{k}{4\pi}\right)^{1/3}$.

Wrong interpretation of the questions

Correct answer

Let r be the radius of base of circular cylinder and h be its height. Let V be volume and S be the total surface area.

$\therefore\ V = \pi r^2 h \quad \Rightarrow \quad h = \dfrac{V}{\pi r^2} \qquad \ldots(1)$

Also, $S = 2\pi rh + \pi r^2 \qquad [\because$ cylinder is open at top$]$

$= 2\pi r . \dfrac{V}{\pi r^2} + \pi r^2 \qquad [\because$ of $(1)]$

$\therefore\ S = \dfrac{2V}{r} + \pi r^2 \quad \Rightarrow \quad \dfrac{dS}{dr} = -\dfrac{2V}{r^2} + 2\pi r$

Now, $\dfrac{dS}{dr} = 0 \quad \Rightarrow \quad -\dfrac{2V}{r^2} + 2\pi r = 0 \quad \Rightarrow \quad 2V = 2\pi r^3$

$\Rightarrow \quad 2\pi r^2 h = 2\pi r^3 \quad \Rightarrow \quad r = h \qquad [\because V = \pi r^2 h]$

Now, $\dfrac{d^2S}{dr^2} = \dfrac{4V}{h^3} + 2\pi$

When $r = h$, $\dfrac{d^2S}{dr^2} = \dfrac{4V}{h^3} + 2\pi > 0$

$\therefore$ S is minimum when $h = r$ i.e. height $=$ radius of base.

TOPIC	QUESTION
RATE OF CHANGE	The side of a square is increasing at the rate of 0.02cm/s .What is the rate at which the area is increasing when the side of the square is 6cm.

Solution done by the students

Given $\frac{da}{dt}$ =0.02 WKT Area of a square =a^2

$\frac{dA}{dt}$ = 2a = 2x 6 =12 sq.cm/s

Error

Missing the concept with respect to time.

Correct answer

Given $\frac{da}{dt}$ =0.02 WKT Area of a square $=a^2$

$\frac{dA}{dt}$ = 2a$\frac{da}{dt}$ = 2x 6 x.02=0.24 sq.cm/s

Practice problems

Q1. The cost function of a firm is given by C = $3x^2 + 2x - 3$. Find the marginal cost when x = 3.

Q2. A particle moves along the curve $y = \frac{2}{3}x^3 + 1$Find the points curve at which the y- coordinate changes as fast as the x- coordinate.

Q3. The radius of a spherical air bubble is increasing at the rate of 0.5 cm/sec. At what rate is the volume of the bubble increasing when its radius is 1cm ?

TOPIC	QUESTION
MAXIMA AND MINIMA	. Find the absolute maximum value and the absolute minimum value forthe function $f(x) = 4x - \frac{x^2}{2}$ in the given interval $x \in \left[-2, \frac{9}{2}\right]$.

SOLUTION DONE BY STUDENTS

f'(x) = 4-x maximum value =4

Error vocabulary difference between value and point

Correct solution ; f(4)=8 f(-2) =-1o f(9/2)= 7.875 absolute maximum value 8 and absolute minimum value -10.

Practice problems.

Find the points of local maximum values and local minimum values if any, for the function f(x) = $3x^4 + 4x^3 - 12x^2 + 12$.

Q. Show that the height of a closed right circular cylinder of given surface and maximum volume is equal to the diameter of the base.

A. S= $2\pi r(h+r)$ and V = $\pi r^2 h$

$\frac{S-2\pi r^2}{2\pi r}$ = h

V= $\pi r^2 \frac{S-2\pi r^2}{2\pi r}$

V= $r\frac{S-2\pi r^2}{2}$

= $\frac{Sr-2\pi r^3}{2}$

$\frac{dV}{dx}=\frac{1}{2}(-6\,\pi r^2)$ or $\frac{dV}{dx}=\frac{1}{2}\left(\frac{dS}{dr}r+S-6\,\pi r^2\right)$

Error: s is considered as variable or derivative of a constant is zero.

Correct Answer

$$\frac{dV}{dx}=\frac{1}{2}(S-6\,\pi r^2)$$

Strategy:

Identification of constants and variables in the function and use of product rule and quotient rule.

<u>Finding critical points</u>

Find the intervals in which the function y = $\frac{1}{x}$ is increasing and decreasing.

Ans: $\frac{dy}{dx}$ = $\frac{-1}{x^2}$

For critical points $\frac{dy}{dx}=0$

$\frac{-1}{x^2}=0$

x^2 = -1

x = ± 1

Misconception: critical points are the points at which derivative is zero.

correct Answer: critical point is x = 0

strategy: need to clear meaning of critical point (points at which either derivative is zero or not exist)

INTEGRALS

CONCEPT MAPPING

(1) $\frac{d}{dx}F(x) = f(x)$, *Then* we write $\int f(x)dx = F(x) + c$. These integrals are called indefinite integrals; C is called a constant of integration.

(2) Some properties of indefinite integrals

(a) The process of differentiation and integration are inverse of each other, $\frac{d}{dx}\int f(x)dx = f(x)$ and $\int f'(x)dx = f(x) + C$, where C is any arbitrary constant

(b) Two indefinite integrals with the same derivatives lead to the same family of curves and so they are equivalent. So it f and g are two functions such that $\frac{d}{dx}\int f(x)dx = \frac{d}{dx}\int g(x)dx$ .Then $\int f(x)dx$ and $\int g(x)dx$ are equivalent.

(c) The integral of the sum of two functions equal the sum of the integrals of the functions i.e., $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

(d) A constant factor may be written either before or after the integrals sign i.e., $\int af(x)dx = a\int f(x)dx$, where a is a constant.

(3) Methods of integrations .

(a) Integration by substitution

(b) Integration using partial fractions

(c) Integration by parts

(4) Definite integrals –

The definite integral is denoted by $\int_a^b f(x)dx$, where a is the lower limit of the integral and b is the upper limit of the integral .The definite integral is evaluated in the following two steps

(a) The definite integral as the limit of a sum

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+f(a+2h)+.........f(a+(n-1)h)]$$

$[or]$

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+.........f(a+(n-1)h)]$$

Where $h=\frac{b-a}{n}\to 0$ as $n\to 0$

(5) Fundamental Theorem of Calculus.

(a) Area function : The function $A(x)=\int_a^b f(x)dx$

(b) Fundamental theorem of integral Calculus – Let f be a Continuous function defined on the closed interval [a ,b] and F be an antiderivative of f .

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b)-F(a)$$

(6) Some properties of Definite Integrals.

$P_0 : \int_a^b f(x)dx = \int_a^b f(t)dt$

$p_1 : \int_a^b f(x)dx = -\int_b^a f(x)dx,$

In particular $\int_a^a f(x)dx = 0$

$$P_2 : \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

$$P_3 : \int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$$

$$P_4 : \int_0^a f(x)dx = \int_0^a f(a-x)dx.$$

$$P_5 : \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$$

$$P_6 : \int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, if & f(2a-x)=f(x) \\ 0, if & f(2a-x)=-f(x). \end{cases}$$

$$P_7 : (i)\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

If f is a even function i.e., $f(-x)=f(x)$

$(ii)\int_{-a}^a f(x)dx = 0$ If f is a odd function i.e., $f(-x)=-f(x)$

TOPIC: INTEGRALS

LEVEL 1

INTEGRATION UNDER STANDARD TYPE

Integrate the following:

1. $\int \frac{dt}{\sqrt{1+9t^2}}$ 2. $\int \frac{dx}{4+9x^2}$ 3. $\int \frac{dx}{\sqrt{4-x^2}}$ 4. $\int \frac{dx}{9x^2-4}$ 5. $\int \frac{dx}{\sqrt{x^2-25}}$ 6. $\int \frac{dx}{\sqrt{x^2+16}}$

INTEGRATION BY PARTIAL FRACTIONS

Integrate the following functions:

1. $\frac{x^2}{(x-1)(x-2)(x-3)}$ 2. $\frac{x^2+x+1}{(x+1)^2(x+2)}$ 3. $\frac{x}{(x-1)^2(x+2)}$

4. $\frac{1}{x(logx)(2+logx)}$ 5. $\frac{x^2+2x+8}{(x-1)(x-2)}$

INTEGRATION BY PARTS

1. Evaluate $\int x\, sec^2\, x\, dx$

2. Evaluate $\int \log_{10} x\, dx$

3. $Evaluate \int x\, e^{2x}$ dx
4. Evaluate $\int e^x\,(tanx + sec^2x)\, dx$
5. Evaluate $\int\ tan^{-1}x\ \ dx$

DEFINITE INTEGRALS

Using properties of integrals evaluate the following:

1. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx}+\sqrt{cosx}} dx$, 2. $\int_{-5}^{0} f(x)\, dx$ where $f(x) = |x| + |x+2| + |x+5|$.

3. $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}} dx$, 4. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x} dx$, 5. $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x\, dx$

LEVEL 2

INTEGRATION UNDER STANDARD TYPE

1. $\int \frac{e^{2x}\,dx}{1+e^{4x}}$, 2. $\int \frac{e^{x}dx}{1+e^{2x}}$, 3. $\int \frac{dx}{5-8x-x^2}$, 4. $\int \frac{\cos x\,dx}{\sqrt{8-\sin^2 x}}$, 5. $\int \frac{dx}{\sqrt{x^2+4x+8}}$,
6. $\int \frac{dx}{x^2+4x+8}$

INTEGRATION BY PARTIAL FRACTIONS

Integrate the following functions:

1. $\frac{\cos x}{(1-\sin x)^3(2+\sin x)}$ 2. $\frac{1}{x(x^4-1)}$ 3. $\frac{x^4}{(x-1)(x^2+1)}$

4. $\frac{3x+5}{x^3-x^2-x+1}$ 5. $\frac{x^3+x+1}{x^2-1}$

INTEGRATION BY PARTS

1. Evaluate $\int (e^{\log x} + \sin x)\cos x\, dx$

2. Evaluate $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\, dx$

3. Evaluate $\int (\sin^{-1} x)^2\ dx$

4. Evaluate $\int\ e^x \cos x\, dx$

DEFINITE INTEGRALS

1. $\int_0^{\frac{\pi}{4}} \log\,(1 + \tan x)dx$

2. $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

3. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\, dx$

4. $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}dx$

5. $\int_0^{\frac{\pi}{2}} \sin 2x.\log(\tan x)dx$

LEVEL 3

INTEGRATION UNDER STANDARD TYPE

1. $\int \frac{2^X dx}{\sqrt{1-4^x}}$
2. $\int \frac{3x\, dx}{1+2x^4}$
3. $\int \frac{x^2\, dx}{1-x^6}$
4. $\int \frac{dx}{1-6x-9x^2}$
5. $\int \sqrt{\sec x - 1}\ dx$
6. $\int \frac{dx}{\sqrt{1-e^{2x}}}$

INTEGRATION BY PARTIAL FRACTIONS

Concepts & Techniques:

A rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}, where\ Q(x) \neq 0.$

If the degree of P(x) is less than the degree of $Q(x)$, then the rational function is called Proper rational function.

If the degree of P(x) is greater than or equal to the degree of $Q(x)$, then the rational function is called Improper rational function.

An improper rational function can be reduced to the proper rational function by long division. i.e., $\frac{P(x)}{Q(x)} = T(x) +$ $\frac{P_1(x)}{Q(x)}, where\ T(x) is\ a\ polynomial\ in\ x\ and\ \frac{P_1(x)}{Q(x)}\ is\ a\ proper\ rational\ function.$

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$ where x^2+bx+c cannot be factorised

LEVEL I

Integrate the following functions:

1. $\frac{x^2}{(x-1)(x-2)(x-3)}$
2. $\frac{x^2+x+1}{(x+1)^2(x+2)}$
3. $\frac{x}{(x-1)^2(x+2)}$
4. $\frac{1}{x(logx)(2+logx)}$
5. $\frac{x^2+2x+8}{(x-1)(x-2)}$

LEVEL II

Integrate the following functions:

1 $\frac{cosx}{(1-sinx)^3(2+sinx)}$

2. $\frac{1}{x(x^4-1)}$

3. $\frac{x^4}{(x-1)(x^2+1)}$

4. $\frac{3x+5}{x^3-x^2-x+1}$

5. $\frac{x^3+x+1}{x^2-1}$

LEVEL III

Integrate the following functions:

1. $\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}$
2. $\frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)}$
3. $\frac{(x+1)}{x(1+xe^x)^2}$
4. $\frac{tan\theta+tan^3\theta}{1+tan^3\theta}$
5. $\frac{1}{sinx-sin2x}$

Integrate the following functions:

1 $\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}$

2. $\frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)}$

3. $\frac{(x+1)}{x(1+xe^x)^2}$

4. $\frac{tan\theta+tan^3\theta}{1+tan^3\theta}$

5. $\frac{1}{sinx-sin2x}$

INTEGRATION BY PARTS

1. Evaluate $\int tan\left(log\frac{x}{3}\right)dx$
2. Evaluate $\int x\ log\left(1+\frac{1}{x}\right)dx$
3. Evaluate $\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$
4. Evaluate $\int\ sin^{-1}\sqrt{\frac{x}{1+x}}\ dx$
5. Evaluate $\int\left(\frac{logx}{(1+logx)^2}\right)dx$

DEFINITE INTEGRALS

1. $\int_{-1}^{\frac{3}{2}}|\ x\sin\pi x\ |\ dx$

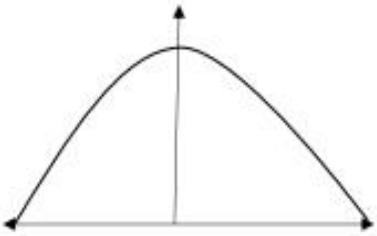
2. $\int_{0}^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

3. $\int_0^{\pi} \log(1-\cos x)\, dx$

4. $\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x + 4\sin^2 x}\, dx$

5. Prove that $\int_0^{a} f(a-x)dx = \int_0^{a} f(x)dx$. Use it to evaluate $\int_0^{2} x\sqrt{2-x}dx$.

CASE STUDY 5:



The bridge connects two hills 100 feet apart. The arch on the bridge is in a parabolic form. The highest point on the bridge is 10 feet above the road at the middle of the bridge as seen in the figure.

Based on the information given above, answer the following questions:

1. The equation of the parabola designed on the bridge is

a. $x^2 = 250y$

b. $x^2 = -250y$

c. $y^2 = 250x$

d. $y^2 = 250y$

2. The value of the integral $\int_{-50}^{50} \frac{x^2}{250} dx$ is

a. $\frac{1000}{3}$

b. $\frac{250}{3}$

c. 1200

d. 0

3. The integrand of the integral $\int_{-50}^{50} x^2 dx$ is ________ function.

a. Even

b. Odd

c. Neither odd nor even

d. None

4. The area formed by the curve $x^2 = 250y$, *x*-axis, *y* = 0 and *y* = 10 is

a. $\frac{1000\sqrt{2}}{3}$

b. $\frac{4}{3}$

c. $\frac{1000}{3}$

d. 0

5. The area formed between $x^2 = 250y$, *y*-axis , *y* = 2 and *y* = 4 is

a. $\frac{1000}{3}$

b. 0

c. $\frac{1000\sqrt{2}}{3}$

d. none of these

ANSWERS

1. b) $x^2 = -250y$

2. a) $\frac{1000}{3}$

3. a) Even

4. c) $\frac{1000}{3}$

5. d) none of these

Identification of error

I Forgetting to put the constant of integration

Evaluate $\int(4x^2+7x-9)dx$

Wrong answer by student: $\int(4x^2+7x-9)dx = \frac{4x^3}{3}+\frac{7x^2}{2}-9x$

Correct answer with explanation: $\int(4x^2+7x-9)dx = \frac{4x^3}{3}+\frac{7x^2}{2}-9x+C$

- **Follow up exercise with answers:**
  Evaluate the following integrals:

1. $\int(\sqrt{x}-\frac{1}{\sqrt{x}})^2dx = \int(x-2+\frac{1}{x})dx = \frac{x^2}{2}-2x+log|x|+C$
2. $\int\frac{dx}{x} = log|x|+C$
3. $\int\frac{x^3\sin(\tan^{-1}x^4)}{1+x^8}dx$ Let $\tan^{-1}x^4 = t. Then, \frac{4x^3}{1+x^8}dx = dt$

$$\int\frac{x^3\sin(\tan^{-1}x^4)}{1+x^8}dx = \int\frac{1}{4}sintdt = -\frac{1}{4}cost+C$$
$$= -\frac{1}{4}\cos(\tan^{-1}x^4)+C$$

4. $\int\frac{dx}{1-x} = -log|1-x|+C$
5. $\int\frac{(logx)^5}{x}dx.$ Let $logx = t.\ \ Then, \frac{1}{x}dx = dt$
   $\int\frac{(logx)^5}{x}dx = \int t^5dt = \frac{t^6}{6}+C = \frac{(logx)^6}{6}+C$

II Mixing differentiation with integration

Evaluate $\int sinx\ dx$

Wrong answer by student : $\int sinx\ dx$ = cosx + C

Correct answer : $\int sinx\ dx$ = - cosx + C

- **Follow up exercise with answers :**
  Evaluate the following integrals :

1. $\int logx\ dx = \int(\log\ x).1\ dx = \log x\int 1\ dx - \int\frac{1}{x}\times x\ dx =$
   $x\log x - x + C$
2. $\int\tan x\ dx = \int\frac{sinx}{cosx}\ dx$ Let cos x = t - sinx dx = dt
   $= \int\frac{-dt}{t} = -\log|t|+C = -\log[cosx]+C$
3. $\int$ x sinx dx = x $\int\sin x\ dx - \int -cosx\ dx = -x\ cosx + sinx + C$
4. $\int\sqrt{x}\ dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}+C$

5. $\int e^x x^2\, dx = x^2 \int e^x\ dx - \int 2xe^x dx = x^2e^x - (2x \int e^x\ dx - \int 2e^x\ dx)$

$$= x^2e^x - 2x\, e^x + 2\, e^x + \mathrm{C}$$

Example:1 $Find\ \int_0^{2\pi} Sin\ x\, dx$

2. $Find\ \int_0^{2\pi} cos\ x\, dx$

COMMON ERRORS

1. $\int \boldsymbol{sinx\ dx} = -\boldsymbol{cosx}$. Instead of $-\boldsymbol{cosx} + \mathrm{c}$.

Tips: Give more examples, like $\frac{d}{dx}(\sin x) = \cos x,$ also

$\frac{d}{dx}(\sin x + c) = \cos x$ where C is any arbitrary constant

2. Considering $\int \boldsymbol{\frac{1}{x}\ dx}$ as $\int \boldsymbol{x^{-1}\ dx}$

Tips:- $\frac{d}{dx}(\log x + c) = \frac{1}{x}$

3. Degree of the numerator must be less than the degree

of the denominator. (e.g) $\int \boldsymbol{\frac{5x^2}{x^2+4x+3}}$ dx ,

Divide and reduce the degree of the numerator.

4. Applying wrong formulae.

(e.g.) $\int \sqrt{\boldsymbol{x^2 - a^2}}$ dx and $\int \sqrt{\boldsymbol{a^2 - x^2}}$ dx etc.

5. In applying $\int \boldsymbol{x^2 - a^2\ dx}$, $\int \sqrt{\boldsymbol{a^2 - x^2}}$ dx,

$\int \boldsymbol{ax^2 + bx + c\ dx}$ etc.,

Tips: the co-efficient of $\boldsymbol{x^2}$ must be unity.

6. They are not able to select which one is a function and

which one is its derivatives.

$\int e^x(\tan x + \log|\sec x|)dx$ Errors $f(x) = \tan x,\ f'(x) = \log|\sec x|$

7. They integrate $Sinbx$ instead of differentiating it

$$\int e^{ax} \sin bx dx = \sin bx.\frac{e^{ax}}{a} + \int \frac{\cos bx}{b}\frac{e^{ax}}{a} dx$$

8. They are not able to reduce inverse trigonometric

function by using substitution.

$\int \frac{x^2(\sin^{-1} x^3)^2}{\sqrt{1-x^6}} dx$ Incorrect substitutions $x = \sin\theta$ or $x^3 = \sin\theta$

Correct substitution $\sin^{-1} x^3 = t$

9. They are applying a lengthy process and take more time.

$$\int \frac{x^2+x+4}{x^2+3x+9} dx = \int \frac{x^2}{x^2+3x+9} dx + \int \frac{x}{x^2+3x+9} dx + \int \frac{4}{x^2+3x+9} dx$$

10. They are not able to identify first function and second

function in integration by parts .

Use the Conviction ILATE

11. When to apply properties of Definite Integrals (e.g.)

$\int_0^1 \boldsymbol{x\,(tan^{-1}x)^2\,dx}$, $\int_0^{\pi} \frac{\boldsymbol{x\tan x}}{\boldsymbol{\sec x + \tan x}}$ dx. , $\int_0^{2\pi} \frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{e^{\sin x}}}$ dx.

Q1 $\int \frac{dx}{x^2+1}$

Ans: Type equation here. $\log(x^2+1)$ +C

Misconception: Using wrong formula $\int \frac{dx}{x}$ = logx + C instead of $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ +C

Correct Solution : $\tan^{-1} x + C$

Q2 $\int \frac{dx}{x^2-a^2}$

Ans: $\frac{1}{2a}\log\frac{(x-a)}{(x+a)}$ +C

Misconception: Using wrong formula $\int \frac{dx}{x^2-a^2}= \frac{1}{2a}\log\frac{(x-a)}{(x+a)}$ +C instead of

$\int \frac{dx}{x^2-a^2}= \frac{1}{2a}\log\left|\frac{(x-a)}{(x+a)}\right|$+C

Correct Solution : $\int \frac{dx}{x^2-a^2}= \frac{1}{2a}\log\left|\frac{(x-a)}{(x+a)}\right|$+C

Q3. $\int \frac{e^x(x^2+1)\,dx}{(x+1)^2}$

Ans: $\int \frac{e^x(x^2+2x+1-2x)\,dx}{(x+1)^2}$

$\int \frac{e^x[(x+1)^2-2x]\ \ dx}{(x+1)^2}$

$\int [1-\frac{2x}{(x+1)^2}]\,e^x$dx

Now child will not be able to apply$\int e^x$[f(x)+f'(x)]= e^xf(x)+C

Misconception: Child understands that changing the numerator,with respect to denominator.

Correct Solution: $\int \frac{e^x(x^2-1+1+1)\,dx}{(x+1)^2}$

$\int e^x[\ \frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}]$ dx

$\int e^x[\ \frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}]$ dx

Now child will be able to apply$\int e^x$[f(x)+f'(x)]= e^xf(x)+C

Ans: $\int e^x[\ \frac{x^2-1}{(x+1)^2}$+C

Q4 $\int xe^{x}$ dx

Ans: $e^{x}\int xdx-\int[\frac{d}{dx}\quad e^{x}\int x$dx]dx

$e^{x}\frac{x^{2}}{2}$ - $\int e^{x}\frac{x^{2}}{2}dx$

Now child will not be able to integrate By parts,2because he has not properly chosen first and second function usind ILATE rule.

Misconception: Child must know the ILATE rule for chosing first and second function, then only integration by parts can be done.

Correct Solution: x $\int e^{x}dx-\int[\frac{d}{dx}\quad x\int e^{x}$dx]dx

$e^{x}x$ - $\int e^{x}dx$

$e^{x}x$ - e^{x}+C

Q5$\int\frac{dx}{(x^{2}+3)((x^{2}+4)}$

Ans: $\frac{1}{(x^{2}+3)((x^{2}+4)}$ =$\frac{ax+b}{(x^{2}+3)}$+$\frac{cx+d}{(x^{2}+4)}$

Now student will feel difficulty in finding values of a,b,c,d.

Misconception: Child must know how to obtain partial fraction if degrees of polynomials in Denominator are equal,then put $x^{2}=y$.

Correct Solution: $\frac{1}{(x^{2}+3)((x^{2}+4)}=\frac{1}{(y+3)(y+4)}$

$$\frac{1}{(y+3)(y+4)}=\frac{A}{(y+3)}+\frac{B}{(y+4)}$$

Here 1=A(y+4)+B(y+3)

Put y= - 4,y= - 3 both sides we get A=1,B= - 1

$$\frac{1}{(y+3)} - \frac{1}{(y+4)}$$

Here, student directly integrate in y without substituting y=x^2

$\frac{1}{(y+3)} - \frac{1}{(y+4)}$ = $\frac{1}{(x^2+3)}$ - $\frac{1}{(x^2+4)}$

Required integration will be

$\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$ - $\frac{1}{2}\tan^{-1}\frac{x}{2}$ + C

<u>Important non-routine problems</u>

<u>Q1.</u> $\int_0^{\sqrt{2}}[x^2]$dx

Q.2 INTEGRATE : $\int\sqrt{tanx}$dx

Q.3 integrate : $\int\frac{sinx+cosx}{\sqrt{1+sin2x}}$dx

Q.4 integrat: $\int_0^{\pi}\frac{xdx}{a^2cos^2x+b^2sin^2x}$dx

.5 integrat: $\int_0^{\frac{\pi}{2}} logsinx$dx

TIPS AND TECHNIQUES ON INTEGRATION :-

(i) When the differentiation of one function is another try to do by substation method.

(ii) When the degree of numerator is greater or equal to the degree of the denominator divide and reduce the degree of the numerator less than the degree of the denominator.

(iii) When the denominator of integration is factorizable try to do by partial fraction.

(iv) When an integration involves two different functions, try to do by parts

(v) In definite integrals, try to apply the properties of the definite integral first.

(vi) $\int_0^4 \left(|x-1|+|x-2|+|x-3|\right)dx$

TIPS :- Open intervals

$$(0\ ,\ 1) \Rightarrow f(x) = -(x-1)-(x-2)-(x-3) = -3x+6$$
$$(1\ ,\ 2) \Rightarrow f(x) = (x-1)-(x-2)-(x-3) = -x+4$$
$$(2\ ,\ 3) \Rightarrow f(x) = (x-1)+(x-2)-(x-3) = x$$
$$(3\ ,\ 4) \Rightarrow f(x) = (x-1)+(x-2)+(x-3) = 3x-6$$

$$I = \int_0^1 (-3x+6)dx + \int_1^2 (-x+4)dx + \int_2^3 (x)dx + \int_3^4 (3x-6)dx$$

(vii) $\int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{9+16\sin 2x}dx$

TIPS :- $\cos^2 x + \sin^2 x - 2\sin x\cos x = 1 - \sin 2x = (\sin x - \cos x)^2$

Hence, convert the Denominator

$$9+16\sin 2x = 9-16(-1+1-2\sin x\cos x) = 25-16(\sin x - \cos x)^2$$

Put $(\sin x - \cos x) = t$ **then** $(\cos x + \sin x)dx = dt$

(vii) $\int \frac{dx}{1+x^4}$

TIPS :- Numerator and Denominator are divided by x^2

$$\int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \frac{1}{2}\int \frac{\left(\frac{1}{x^2}+1\right)-\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}} = \frac{1}{2}\left[\int \frac{\left(1+\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}dx\right]-\frac{1}{2}\left[\int \frac{\left(1-\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}dx\right]$$

If numerator $\left(1+\frac{1}{x^2}\right)$ then put $\left(x-\frac{1}{x}\right)=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$ and convert the denominator $\left(x^2+\frac{1}{x^2}\right)=\left(x-\frac{1}{x}\right)^2+2$

If numerator $\left(1-\frac{1}{x^2}\right)$ then put $\left(x+\frac{1}{x}\right)=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$ and convert the denominator $\left(x^2+\frac{1}{x^2}\right)=\left(x+\frac{1}{x}\right)^2-2$

(ix) $\int \frac{dx}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$

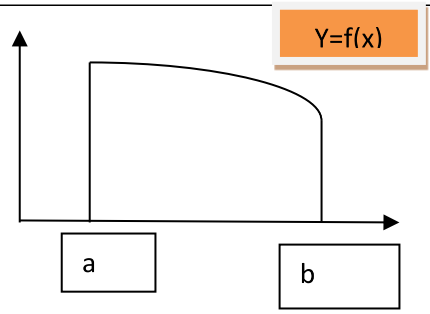
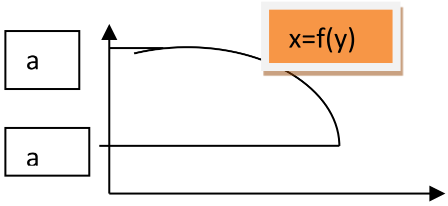
TIPS:- $x^{\frac{1}{6}}=t \Rightarrow x^{\frac{1}{2}}=t^3, x^{\frac{1}{3}}=t^2 \Rightarrow dx = 6t^5dt$

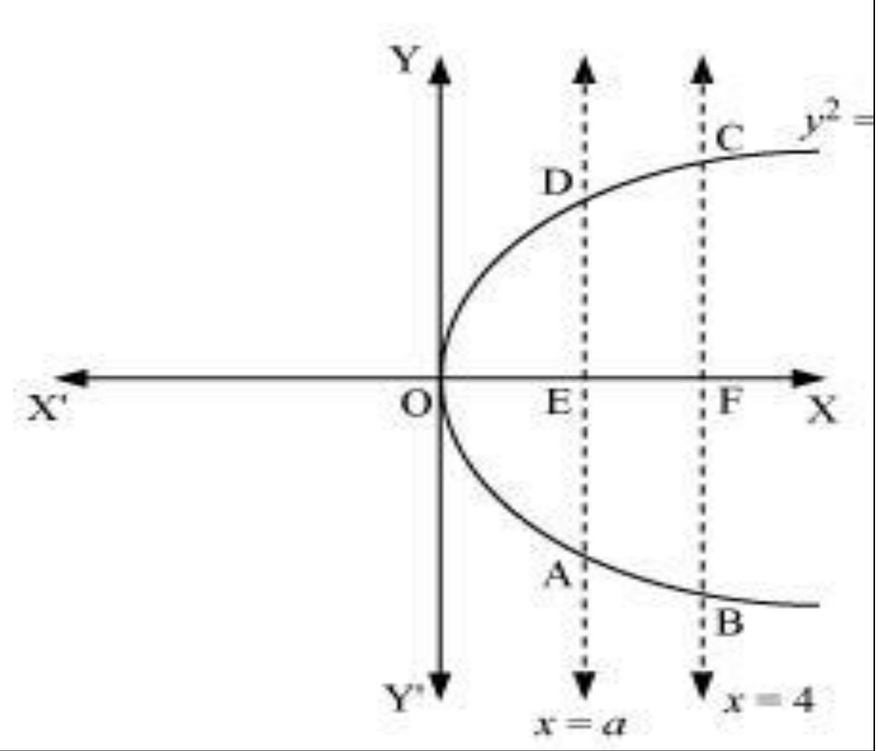
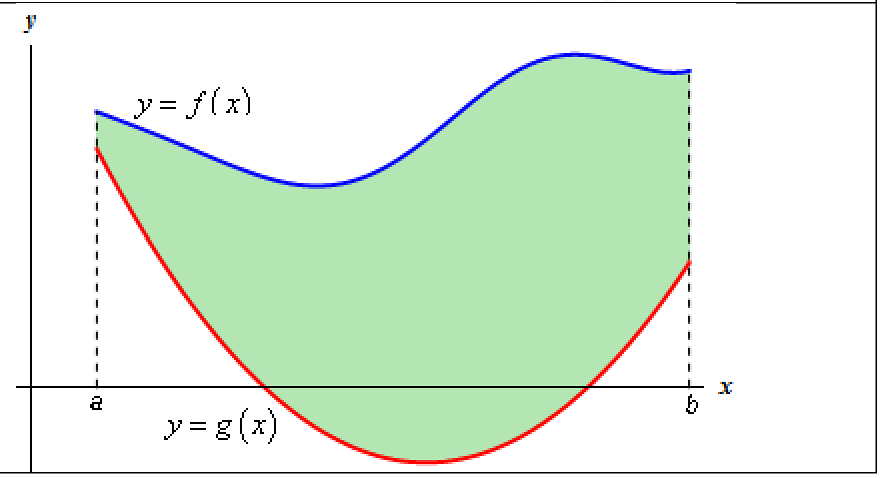
(x) $\int \frac{dx}{e^x-1}$

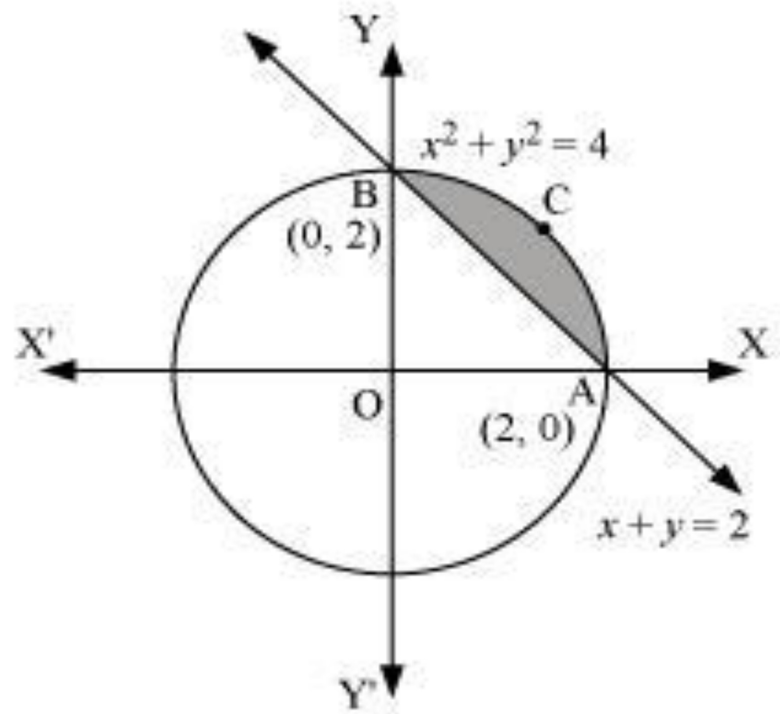
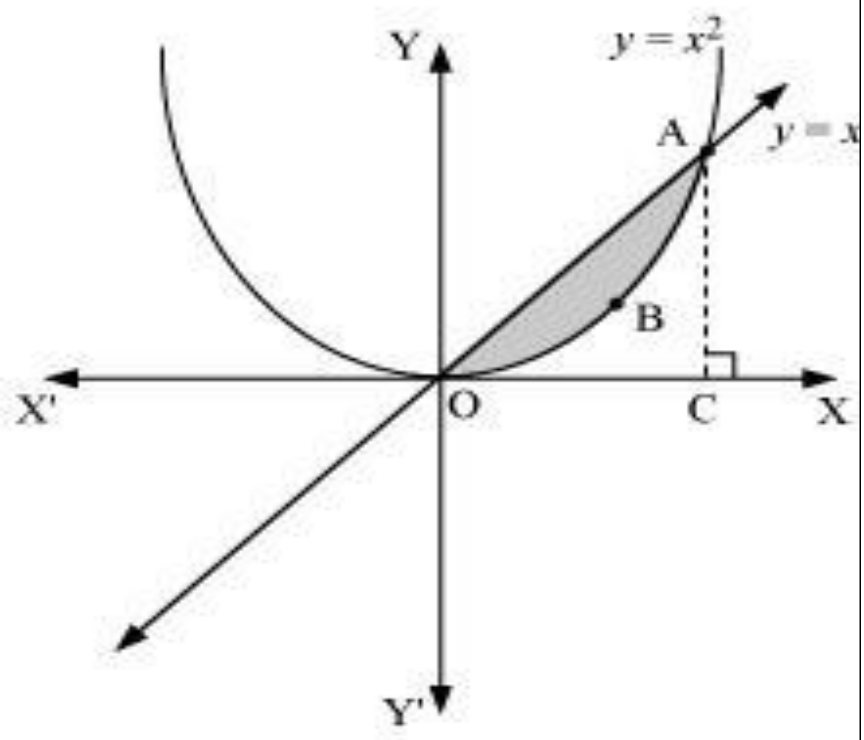
TIPS :-Multiply the numerator and denominator by e^x **and put** $e^x=t$

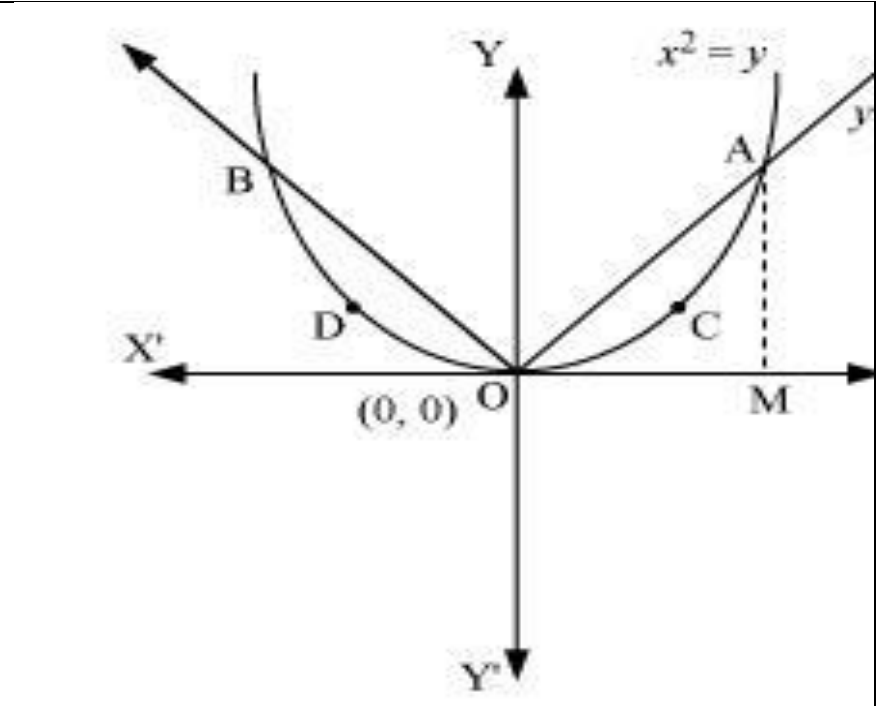
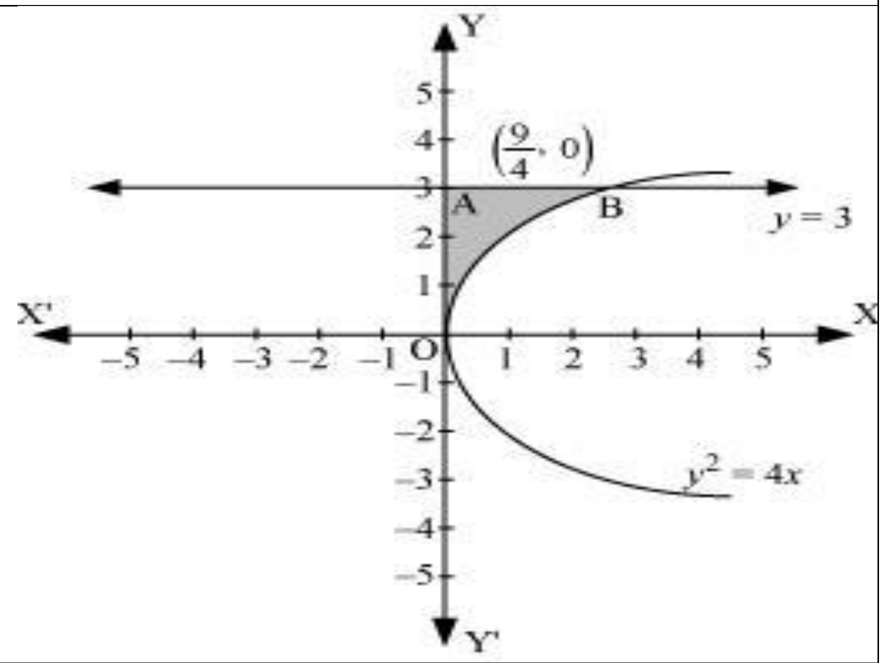
(xi) $\int \frac{dx}{x\left(x^n+1\right)}$

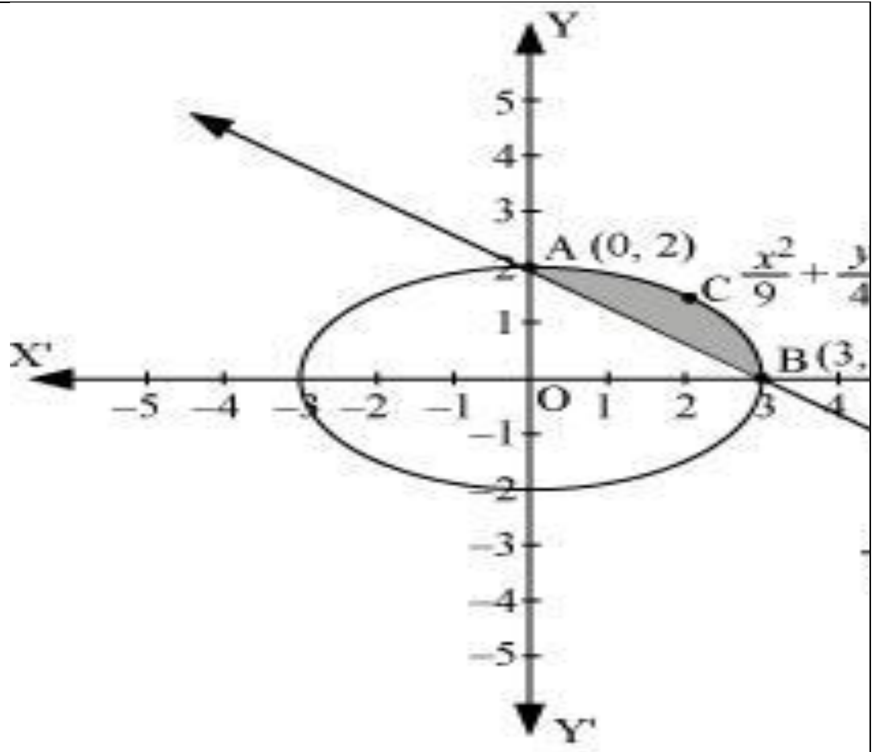
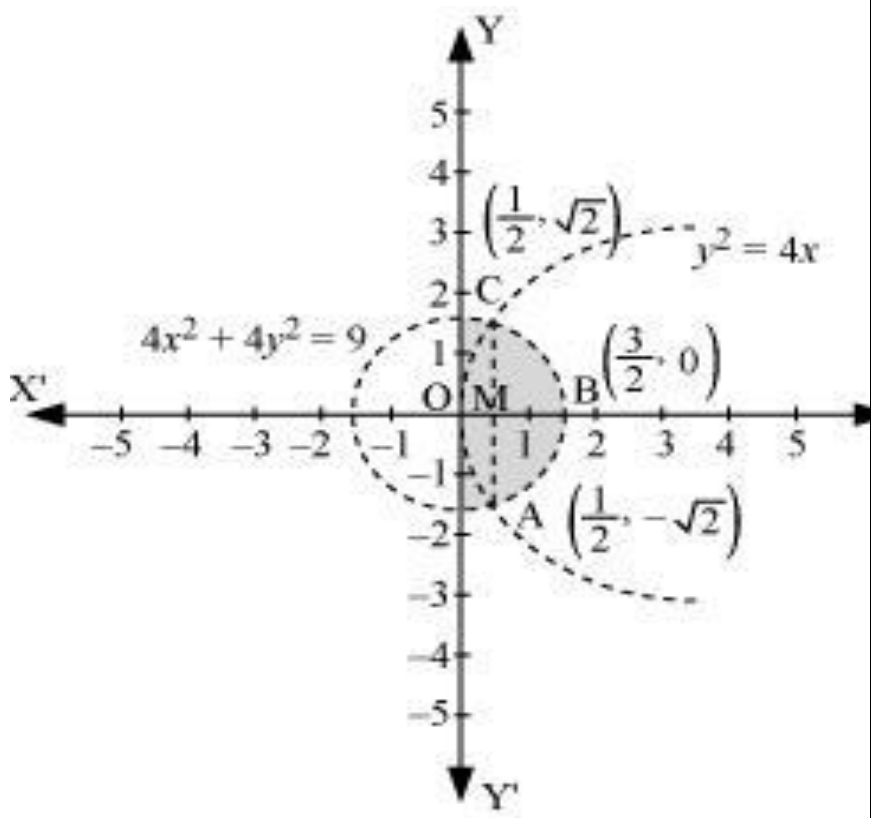
TIPS :- Multiply the numerator and denominator by x^{n-1} **and put** $x^n=t$

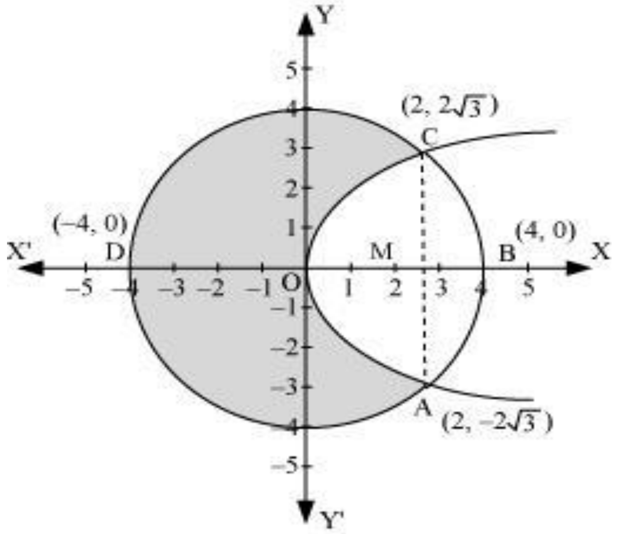
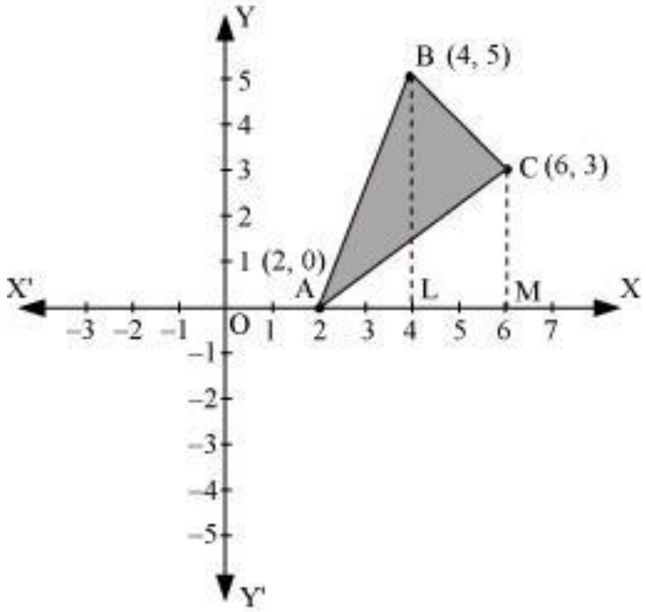
CONCEPT MAPPING APPLICATION OF INTEGRATION	
The area of the region bounded by the curve y=f(x),x-axis and the lines x=a,x=b is given by • A=$\int_a^b ydx$	
The area of the region bounded by the curve x=f(y),y-axis and the lines y=a, y=b is given by A=$\int_a^b x\,dx$	
Steps to find area	STEPS - DRAW THE DIAGRAM - MAKE A SHADED REGION - FIND INTERSECTION POINTS - IDENTIFY THE LIMITS - WRITE THE INTEGRAL(S) FOR THE REGION - EVALUATE THE INTEGRAL - THE VALUE SHOULD BE POSITIVE

- CURVE - Parabola - y^2 =x ,x=a, - x=4	
Two curves are given **A=$\int_a^b f(x) - g(x)dx$**	

Examples Circle and line	
Parabola and line	

Parabola modulus	
Parabola and y=a	

Ellipse and line	
Inequality problem	$\left\{(x,y): y^2 \le 4x, 4x^2+4y^2 \le 9\right\}$ 

Exterior area problem	 Find the area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$
Area of triangle	 Using the method of integration find the area of the triangle ABC, coordinates of whose vertices are A (2, 0), B (4, 5) and C (6, 3

AREA UNDER CURVES USING INTEGRATION

LEVEL 1

1.Find the area of the region bounded by the curves $y^2 = 9x$, $y = 3x$.

2. Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x +$
2

3. Sketch the region $\{(x, 0) : y = \sqrt{4 - x^2}\}$ and x-axis. Find the area of the region
using integration

4. Calcualte the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.

5. Find the area enclosed by the circle $x^2 + y^2 = a^2$

LEVEL 2

6. Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.

7. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2$□□

8. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x axis
and the lines $x = 2$ and $x = 8$.

9.Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

10.Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$

LEVEL 3

□□□Find the area of region bounded by the triangle whose vertices are (–1, 1), (0,5) and (3, 2), using integration.

12. Using integration find the area of the triangular region whose sides have the equations y = 2x + 1, y = 3x + 1 and x = 4..

13.The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

14. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$

Creative and Critical thinking question

An elliptical sized swimming pool is to be constructed in a town whose equation

is $\frac{X^2}{100}+\frac{Y^2}{64}=1$

(i) Find the area of swimming pool as shown in figure using integration.

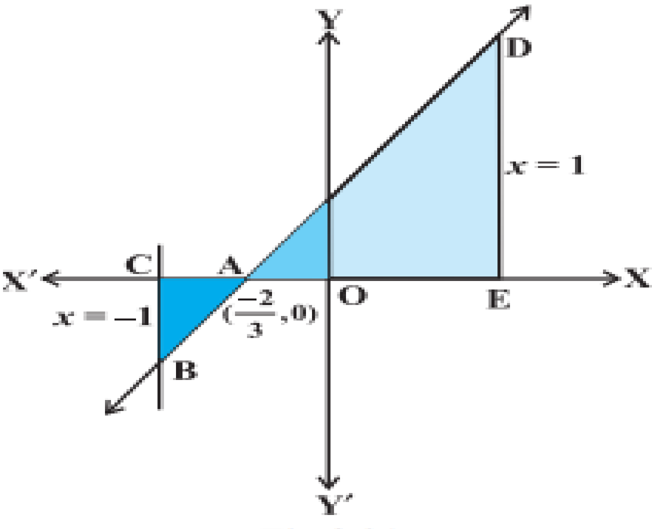
(ii) If the rate of covering its floor with tiles is Rs 100/m^2, then find total cost to cover entire floor of swimming pool.

(iii)) If the depth of the Swimming pool is 6m then What is its capacity?

TOPIC: AREA UNDER THE CURVE

QUESTION: Find the area of the region bounded by the line y=3x +2, the x-axis and the ordinates x=-1 and x=1

Wrong answer



Required area $A = \int_{-1}^{-2/3}(3x + 2)\mathrm{dx} + \int_{-2/3}^{1}(3x + 2)dx$

$$=\left[\frac{3x^2}{2} + 2x\right]_{-1}^{-\frac{2}{3}} + \left[\frac{3x^2}{2} + 2x\right]_{-2/3}^{1}$$

$$= \left(\frac{2}{3} - \frac{4}{3}\right) - \left(\frac{-1}{2}\right) + \frac{7}{2} - \left(\frac{2}{3} - \frac{4}{3}\right)$$

=4 Sq units

Error identified: while finding area under curves ,if the area is below x-axis should take modulus value. Here area between -1 to $\frac{-2}{3}$ is below x- axis .Therefore for that part modulus should be taken

Correct answer:

Required area $A = \left|\int_{-1}^{-2/3}(3x + 2)\mathrm{dx}\right| + \int_{-2/3}^{1}(3x + 2)dx$

$$= \left| \left[\frac{3x^2}{2} + 2x \right]_{-1}^{-\frac{2}{3}} \right| + \left[\frac{3x^2}{2} + 2x \right]_{-2/3}^{1}$$

$$= \left| \left(\frac{2}{3} - \frac{4}{3} \right) - \left(\frac{-1}{2} \right) \right| + \frac{7}{2} - \left(\frac{2}{3} - \frac{4}{3} \right)$$

$$= \left| \frac{-2}{3} + \frac{1}{2} \right| + \frac{7}{2} - \left(\frac{-2}{3} \right)$$

$$= \frac{1}{6} + \frac{25}{6} = \frac{13}{6} \; Sq\ units$$

Common Errors Committed by the students:-

1. Incorrect drawing of the graph for the given curves.
2. Incorrect locating of the required enclosed region.
3. Finding point of intersection incorrectly.
4. Basic integration errors.
5. Area of the region below x-axis.
6. Unit of the area.

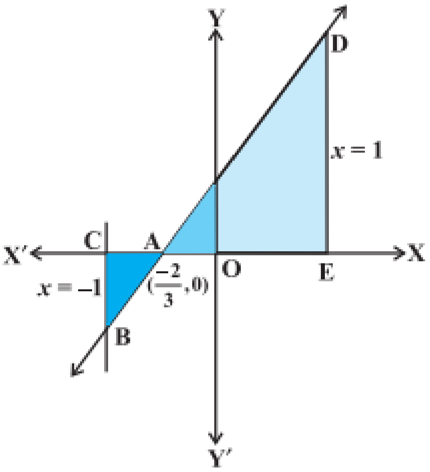
1.Find the area of region bounded by the line y = 3x + 2, the x axis and the ordinates x=-1 and x =1.

Wrong answer :

The required area = Area of the region ACBA + Area of the region ADEA

$$= \int_{-1}^{\frac{-2}{3}} (3x+2)dx + \int_{\frac{-2}{3}}^{1} (3x+2)dx$$

$$= \left[\frac{3x^2}{2} + 2x \right]_{-1}^{\frac{-2}{3}} + \left[\frac{3x^2}{2} + 2x \right]_{\frac{-2}{3}}^{1} = \frac{-1}{6} + \frac{25}{6} = \mathbf{4}$$



MISCONCEPTION : When area is below x- axis while integrating w.r.t x .

Correct answer :

The required area = Area of the region ACBA + Area of the region ADEA

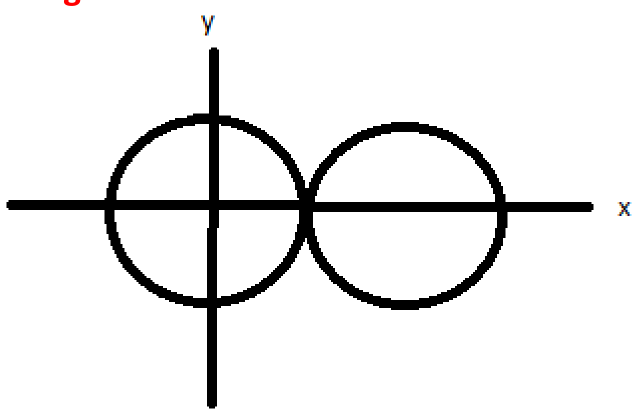
$$= \left|\int_{-1}^{\frac{-2}{3}}(3x+2)dx\right| + \int_{\frac{-2}{3}}^{1}(3x+2)dx$$

$$= \left|\left[\frac{3x^2}{2}+2x\right]_{-1}^{\frac{-2}{3}}\right| + \left[\frac{3x^2}{2}+2x\right]_{\frac{-2}{3}}^{1} = \frac{1}{6}+\frac{25}{6}=\frac{13}{3}$$

Teaching strategy: **When area is below x- axis by integrating w.r.t x, area should be with the correct sign i.e Modulus of that part is to be taken**

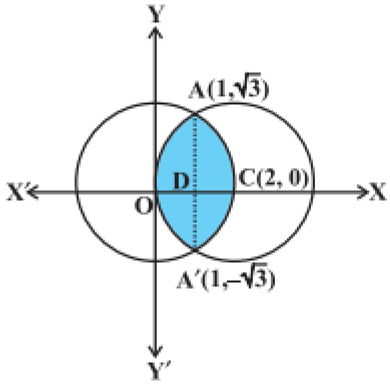
Q. Find the area of the region enclosed between the two circles x^2 +y^2= 4 and $(x-2)^2$ + y^2 = 4.

Wrong answer :



MISCONCEPTION: Student locate wrong sketch of cicle when centre is not at origin

Correct answer



Required area of the enclosed region OACA′O between circles

$= 2$ [area of the region ODCAO] (Why?)

$= 2$ [area of the region ODAO + area of the region DCAD]

$$= 2\left[\int_0^1 y\,dx + \int_1^2 y\,dx\right]$$

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right] \qquad \text{(Why?)}$$

$$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+\ 2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 + \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[\left(-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right)\right) - 4\sin^{-1}(-1)\right] + \left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$$

$$= \left[\left(-\sqrt{3} - 4\times\frac{\pi}{6}\right) + 4\times\frac{\pi}{2}\right] + \left[4\times\frac{\pi}{2} - \sqrt{3} - 4\times\frac{\pi}{6}\right]$$

$$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3}\right)$$

$$= \frac{8\pi}{3} - 2\sqrt{3}$$

Strategy: Give the basic knowledge to draw the figure of circles, when the centre of circle is not at origin.

Q. Find the area of the region bounded by the two parabolas y = x^2 *and*

$y^2 = . x.$

Wrong answer :

The required area is=$\int_0^1 [g(x) - f(x)]\, dx$

MISCONCEPTION : Wrong identification of lower curve and upper curve

Correct answer:

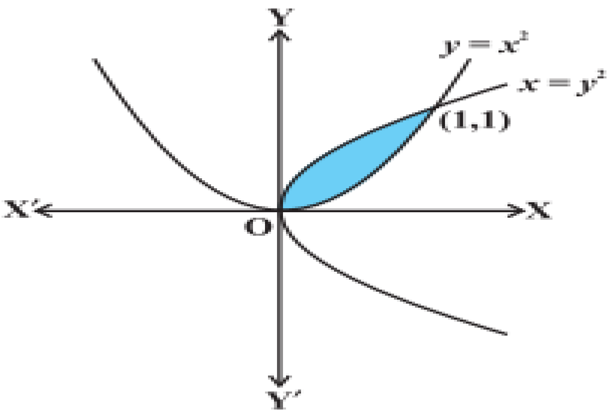


Fig 8.15

Here, we can set $y^2 = x$ or $y = \sqrt{x} = f(x)$ and $y = x^2$ $= g(x)$, where, $f(x) \geq g(x)$ in [0, 1].

Therefore, the required area of the shaded region

$$= \int_0^1 [f(x) - g(x)]\, dx$$

$$= \int_0^1 \left[\sqrt{x} - x^2\right] dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

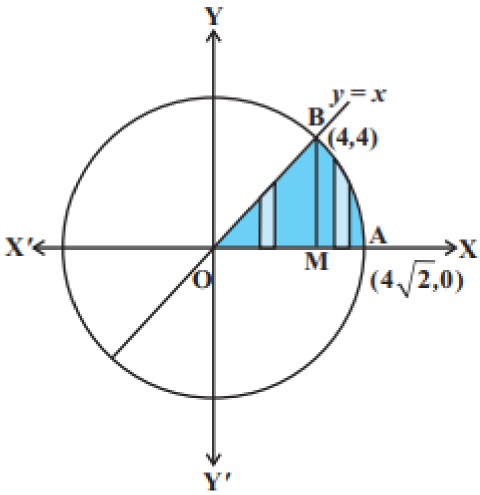
Strategy : **Give the basic knowledge of the question if it is along x-axis th**

Area = $\int_a^b (upper\ curve - lower\ curve)dx$

Q.Find the area of the region in the first quadrant enclosed by the x –axis ,the line y = x , and the circle $x^2 + y^2$ = 32.

wrong answer : **the required area**

$$= \int_0^{4\sqrt{2}} (x + \sqrt{32 - x^2})\, dx$$



Misconception: **Students are unable to decide the correct limits of integration of splited parts of the required area .**

Correct Answer:

Solution The given equations are

$$y = x \quad \dots (1)$$

and $$x^2 + y^2 = 32 \quad \dots (2)$$

Solving (1) and (2), we find that the line and the circle meet at B(4, 4) in the first quadrant in fig . Draw perpendicular BM to the x-axis.

Therefore, the required area = area of the region OBMO + area of the region BMAB.

Now, the area of the region OBMO

$$= \int_0^4 y\,dx = \int_0^4 x\,dx \quad \dots (3)$$

$$= \frac{1}{2}\left[x^2\right]_0^4 = 8$$

Again, the area of the region BMAB

$$= \int_4^{4\sqrt{2}} y\,dx = \int_4^{4\sqrt{2}} \sqrt{32-x^2}\,dx$$

$$= \left[\frac{1}{2}x\sqrt{32-x^2} + \frac{1}{2}\times 32\times \sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left(\frac{1}{2}4\sqrt{2}\times 0 + \frac{1}{2}\times 32\times\sin^{-1}1\right) - \left(\frac{4}{2}\sqrt{32-16} + \frac{1}{2}\times 32\times\sin^{-1}\frac{1}{\sqrt{2}}\right)$$

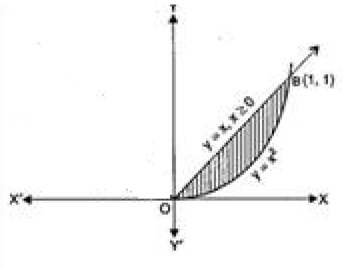
$$= 8\pi - (8+4\pi) = 4\pi - 8 \qquad \text{... (4)}$$

Adding (3) and (4), we get, the required area $= 4\pi$.

Strategy : To give the knowledge to decide the limits of different parts of required area

Q .Find the area of the region bounded by the parabola $y = x^2$ **and y** $=|x|$

Ans. Wrong answer : (by Drawing wrong figure)



Area between parabola $y = x^2$ and $x-$ axis between limits $x=0$ and $x=1$

$$\int_0^1 y\,dx = \int_0^1 x^2\,dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3} \qquad \text{...........(i)}$$

And Area of ray $y = x$ and $x-$ axis,

$$= \int_0^1 y\,dx = \int_0^1 x\,dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2} \qquad \text{...........(ii)}$$

$\therefore$ Required shaded area in first quadrant = Area between ray $y = x$ for $x \geq 0$ and $x-$ axis –

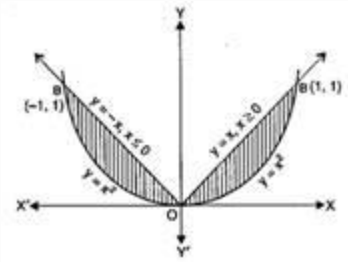
Area between parabola $y = x^2$ and $x-$ axis in first quadrant

= Area given by eq. (ii) – Area given by eq. (i) = $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq. units

Misconception: Students are unable to interpret the definition and graphical representation of $|x|$.

Correct answer:

The required area is the area included between the parabola and the modulus function $y=|x|=\begin{cases} x, & \text{if } x\geq 0 \\ -x, & \text{if } x\leq 0 \end{cases}$



To find: Area between the parabola $y = x^2$ and the ray $y = x$ for $x \geq 0$

Here, Limits of integration $\Rightarrow$ $y=x$

$\Rightarrow x^2 = x \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1)=0 \Rightarrow x=0, x=1$

Now, for $y=|x|$, table of values,

$y=x$ if $x\geq 0$ $y=-x$ if $x\leq 0$

Now, Area between parabola $y=x^2$ and $x-$ axis between limits $x=0$ and $x=1$

$= \int_0^1 y\,dx = \int_0^1 x^2\,dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$ ....(i)

And Area of ray $y=x$ and $x-$ axis,

$= \int_0^1 y\,dx = \int_0^1 x\,dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}$ ....(ii)

$\therefore$ Required shaded area in first quadrant = Area between ray $y=x$ for $x\geq 0$ and $x-$ axis –

Area between parabola $y = x^2$ and $x-$ axis in first quadrant

Area given by eq. (ii) – Area given by eq. (i) $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

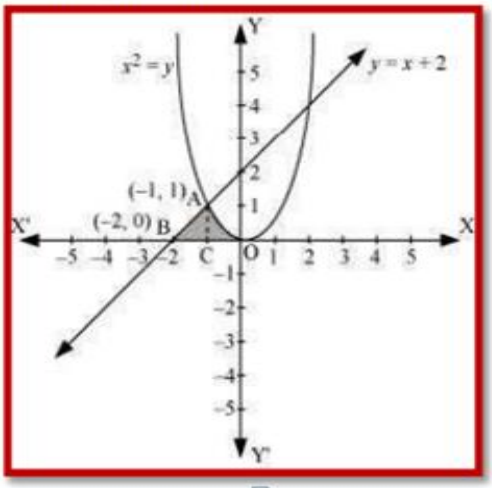
REQUIRED AREA = 2 X $\frac{1}{6}$ = $\frac{1}{3}$ sq. units

Strategy : To give the knowledge of graph of standard form of parabola and modulus function

NON ROUTINE QUESTIONS WITH SOLUTION.

Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x + 2$ and xaxis

The area of the region enclosed by the parabola, $x^2 = y$, the line, $y = x + 2$, and x-axis is represented by the shaded region OABCO as



The point of intersection of the parabola, $x^2 = y$, and the line, $y = x + 2$, is A (−1, 1). ∴

Area OABCO = Area (BCA) + Area COAC

$$= \int_{-2}^{-1}(x+2)dx + \int_{-1}^{0}x^2dx$$

$$= \left[\frac{x^2}{2}+2x\right]_{-2}^{-1} + \left[\frac{x^3}{3}\right]_{-1}^{0}$$

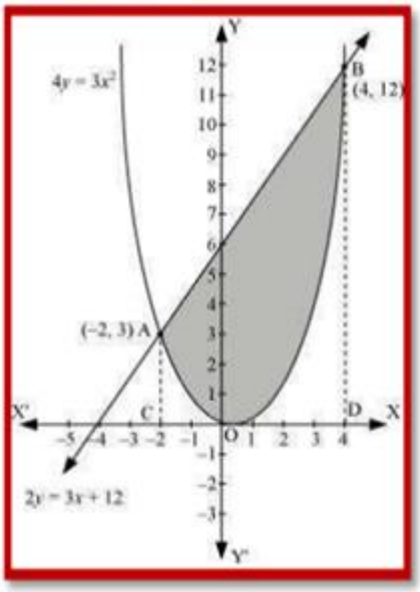
$$= \left[\frac{(-1)^2}{2}+2(-1)-\frac{(-2)^2}{2}-2(-2)\right] + \left[-\frac{(-1)^3}{3}\right]$$

$$= \left[\frac{1}{2}-2-2+4+\frac{1}{3}\right]$$

$$= \frac{5}{6} \text{ units}$$

Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$

The area enclosed between the parabola, $4y = 3x^2$, and the line, $2y = 3x + 12$, is represented by the shaded area OBAO as



The points of intersection of the given curves are A (−2, 3) and (4, 12).

We draw AC and BD perpendicular to x-axis.

∴ Area OBAO = Area CDBA − (Area ODBO + Area OACO)

$$= \int_{-2}^{4} \frac{1}{2}(3x+12)\,dx - \int_{-2}^{4} \frac{3x^2}{4}\,dx$$

$$= \frac{1}{2}\left[\frac{3x^2}{2} + 12x\right]_{-2}^{4} - \frac{3}{4}\left[\frac{x^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{2}[24 + 48 - 6 + 24] - \frac{1}{4}[64 + 8]$$

$$= \frac{1}{2}[90] - \frac{1}{4}[72]$$

$$= 45 - 18$$

$$= 27 \text{ units}$$

Measures to reduce the errors:-

1. Explain graph of the different types of curves like straight line, parabola, circle, ellipse etc. correctly.
2. Explain how to find point of intersection i.e. to find common points of the given curves.
3. To shade the correct region by giving many examples of standard curves.
4. When area is below x-axis, area should be added with the correct sign.

Common Errors Committed by the students:-

S.no.	topic	Errors committed by the students	remedial
1	Finding area through integration	Incorrect drawing of the graph for the given curves.	Explain graph of the different types of curves like straight line, parabola, circle, ellipse etc. correctly.
2	Finding area through integration	Finding point of intersection incorrectly.	Explain how to find point of intersection i.e. to find common points of the given curves.
3	Finding area through integration	Basic errors in process of integration	Basics of integration should be discussed
4	Finding area through integration	Area of the region below x-axis	Negative sign should be given before integrand
5	Finding area through integration	Unit of the area.	Area of unit like sq. unit should be highlighted

DIFFERENTIAL EQUATION

CONCEPT MAPPING:

- Definition of differential equation.
- **Order of a differential equation**

- **Without radical sign**: Order is the order of the highest order derivative appearing in the equation
- Example: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$.
- **With radical sign** : like $\sqrt{\ }$, $\sqrt[3]{\ }$, etc. remove the radical sign by squaring, cubing etc. then Order is the order of the highest order derivative appearing in the equation.
- Example: $\sqrt{1 - \left(\frac{dy}{dx}\right)^2} = \left(\frac{d^2y}{dx^2}\right)^{\frac{1}{3}}$
-
- **Degree of a differential equation**
- The degree of a differential equation is the degree of the highest order derivative occurring in the equation, when the differential coefficients are made free from radicals if each term involving derivatives of differential equation is polynomial(can be expressed as a polynomial).
- Example: $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - sinY' = 0$
-
- **Verifying the given function as a solution of the given differential equation.**

Solving a differential equation:

- Variable separable
- Type-1: $\frac{dy}{dx} = \frac{f(x)}{g(y)}, or\ \frac{g(y)}{f(x)}$ then separate the variables and integrate.
- Type-2: $f_1(x).\, g_1(y)\, dy + f_2(x).\, g_2(y)\, dx = 0$, then divide by $f_1(x).\, g_2(y)$ and integrate.
- Type-3: $\frac{dy}{dx} = f(ax + by + c), substitute\ ax + by + c = t\ and\ solve.$

- Homogeneous

- If the question is asked to prove homogeneous find $f(\lambda x, \lambda y) = \lambda^n f(x, y)$then "f" is homogeneous of degree "n".

- Step 1: put y= vx and $\frac{dy}{dx} = v + x.\frac{dv}{dx}$ where "v" is a function of "x" alone.

- Step 2: The equation reduce to variable separable and solve.

- Some homogeneous equation can be solved by substituting $x = vy$.

- Linear Differential equation
- Type1 : $\frac{dy}{dx} + Py = Q,$
- where P and Q are functions of "x" alone or constants.
- Step 1: Identify P and Q.
- Step 2: Find $\int P\,dx$ and find $e^{\int P\,dx}$(Note: If $\int P\,dx = \log(f(x))$, then $e^{\int P\,dx} = f(x)$).
- Step 3: Solution $y.e^{\int P\,dx} = \int Q.e^{\int P\,dx}\,dx + C$.

- Type2 : $\frac{dx}{dy} + Px = Q,$
- where P and Q are functions of "y" alone or constants.
- Step 1: Identify P and Q.
- Step 2: Find $\int P\,dy$ and find $e^{\int P\,dy}$(Note: If $\int P\,dy = \log(f(y))$, then $e^{\int P\,dy} = f(y)$).
- Step 3: Solution $x.e^{\int P\,dy} = \int Q.e^{\int P\,dy}\,dy + C$.

Find the type of the following differential equations:

- $(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4}$.
- $\sec x\,\frac{dy}{dx} - y = \sin x$.
- $x^2\frac{dy}{dx} = y^2 + 2xy$
- $x\frac{dy}{dx} = y - x\,tan\left(\frac{y}{x}\right)$
- $cos^2 x\,\frac{dy}{dx} + y = \tan x$
- $\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\frac{dy}{dx} = 0$

DIFFERENTIAL EQUATIONS

Q.NO.	LEVEL-1
1.	Solve the differential equation $\frac{dy}{dx} = \frac{-y}{x}$ **xy =c**
2.	Solve the differential equation ydy + xdx =0 $\mathbf{x^2 + y^2}$ =c
3.	Solve: Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}$; $y \neq 2$ 2y-($y^{2/}$/2)= ($x^{2/}$/2)+ x +c
4.	Solve: Find the general solution of the differential equation $\frac{dy}{dx} = \frac{\sqrt{1-y2}}{\sqrt{1-x^2}}$; $x \neq 1$ $\sin^{-1} x - \sin^{-1} y$ =c
5.	Solve: $\frac{dy}{dx} = ye^x$ **log y = $\mathbf{e^x}$ +c**
6.	Solve the following differential equation $\frac{dy}{dx} = \frac{x+y}{x}$ Ans: $y = x\log\lvert x\rvert + Cx$
7.	Solve the following differential equation $y + x\frac{dy}{dx} = x - y\frac{dy}{dx}$ Ans: $y^2 + 2xy - x^2 = C$
8.	Find the particular solution of the differential equation $x\frac{dy}{dx} - y + x\,cosec\left(\frac{y}{x}\right) = 0$; given that $x = 1\ when\ y = 0$ Ans: $\cos\left(\frac{y}{x}\right) = \log\lvert x\rvert + 1$
9.	Solve the following differential equation $x\frac{dy}{dx} = y - x\,tan\left(\frac{y}{x}\right)$ Ans: $\log\left\lvert x\sin\left(\frac{y}{x}\right)\right\rvert = C$
10.	Solve the following differential equation and find the particular solution. $\left(xe^{\frac{x}{y}} + y\right)dx = xdy$; given that $y(1) = 1$ Ans: $y = x - x\log(1 - e\log\lvert x\rvert)$

11.	Write the integrating factor of differenial equation $\frac{dy}{dx} + y\ tanx - secx = 0$ secx
12.	Solve the differenial equation $\frac{dy}{dx} + y = cosx - sinx$ $e^x sinx$ +C
13.	Integrating factor of differential equation $\frac{dy}{dx} + \frac{2y}{x} = e^x$ x^2
14.	Solve $\frac{dy}{dx} cos^2x + y = 1$ $ye^x = e^x cosx + C$
15.	If the integrating factor of the differential equation $\frac{dy}{dx} + Py = Q$ is sinx, then find value of P cotx
	LEVEL-2
1.	Find the particular solution of $\frac{dy}{dx} = -4x^2y$ given that y = 1and x = 1 **log y= --(4 x^3/3) + (4/3)**
2.	Solve the separable differential equation $\frac{dy}{dx} =$ (x - 2)(y^2 - 9), y(0) = -1 **(1/6)log$\frac{\|y-3\|}{\|y+3\|} = \frac{x^2}{2}$-2x +$\frac{1}{6}$ log2**
3.	Find the particular solution of separable differential equations: $\frac{dr}{d\theta} = \frac{r^2}{\theta}$, r(1) = 2. $\frac{-1}{r} = \log\theta - \frac{1}{2}$
4.	Check if the differential equation y' = xy - 21 + 3y - 7x is of variable separable type s **Not of variable separable type**
5.	Solve the differential equation 2 *y dy* = (x^2 + 1) *dx*. $y^2 = \frac{x^3}{3} + x + C$
6.	Show that the following differential equation is homogeneous and then solve it $(x-y)dy = (x+y)dx$

	Ans: $\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\log\lvert x^2+y^2\rvert=C$
7.	Solve the following differential equation $x\,cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\,cos\left(\frac{y}{x}\right)+x$; $(x\neq 0)$ Ans: $sin\left(\frac{y}{x}\right)=\log\lvert x\rvert+C$
8.	Solve the following differential equation $\left[y-x\,cos\left(\frac{y}{x}\right)\right]dy+\left[y\,cos\left(\frac{y}{x}\right)-2x\,sin\left(\frac{y}{x}\right)\right]dx=0$ Ans: $y^2-2x^2sin\left(\frac{y}{x}\right)=C$
9.	Find the particular solution of the differential equation $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\,sin\left(\frac{y}{x}\right)=0$; given that $x=1\ when\ y=\frac{\pi}{2}$ Ans:$cos\left(\frac{y}{x}\right)=\log\lvert x\rvert$
10.	Find the particular solution of the differential $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$ Ans: $x+ye^{\frac{x}{y}}=C$
11.	Which of the following is the general equation of $\frac{dy}{dx}+y\,tanx-secx=0$ a) ysecx=tanx+C b)y tanx=secx+C c) tanx =ytanx+C d) x secx= tany+C a) ysecx=tanx+C.
12.	$Solve\ \ x\frac{dy}{dx}+y=e^x$ $y=\frac{e^x}{x}+\frac{k}{x}$
13.	Solve $xlogx\frac{dy}{dx}+y=2logx$ y= log x + $\frac{c}{\log x}$
14.	Solve $(cos^2x)\frac{dy}{dx}+y=tanx$ $ye^{tanx}=(tanx)e^{tanx}-e^{tanx}+C$
15.	Solve $(1+x^2)\frac{dy}{dx}+y=\tan^{-1}x$ Y = $tan^{-1}x-1+ce^{-\,tan^{-1}x}$

	LEVEL-3
1.	Solve the following equation $\frac{dy}{dx} = (1 + e^{-x})(y^2 - 1)$ $\frac{1}{2} \log \frac{\lVert y+1 \rVert}{\lVert y-1 \rVert}$ =x-$e^x + c$
2.	Solve the differential equation $xydx - (x^2 + 1)\, dy = 0$ $\left(x^2 + 1\right)^2$=Cy
3.	Find particular solution of the following differential equation $y' = \frac{3x^2+4x-4}{2y-4}$ with initial condition y(1)=3 **y^2-4y=x^3+x^2-4x-10**
4.	Find the particular solution of the following differential equation $2xyy'-y^2+x^2 = 0$ where y(1)=2 **$x^2+y^2=3x$**
5.	Solve the DE $xy^2 \frac{dy}{dx}=1-x^2+y^2-x^2y^2$ $y - \tan^{-1} y = \log\lvert x\rvert - \frac{x^2}{2}$+C
6.	Find the particular solution of the differential equation $2ye^{\frac{x}{y}}.dx + \left[y - 2xe^{\frac{x}{y}}\right] dy = 0$ given that $x = 0$ $when\ y = 1$ Ans: $2e^{\frac{x}{y}} + \log\lvert y\rvert = 2$
7.	Solve the following differential equation $x \cos\left(\frac{y}{x}\right)(ydx + xdy) = y \sin\left(\frac{y}{x}\right)(xdy - ydx)$ Ans:$\log\left\lvert\frac{\sec\left(\frac{y}{x}\right)}{xy}\right\rvert = C$
8.	Find the particular solution of the differential equation $(x - y)\frac{dy}{dx} = x + 2y, given\ that\ x = 1; y = 0$ Ans:$\frac{\pi}{2\sqrt{3}} = -\frac{1}{2}\log(x^2 + xy + y^2) + \sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)$
9.	Solve the following differential equation $\left[xsin^2\left(\frac{y}{x}\right) - y\right] dx + xdy = 0$ Ans: $y = x\cot^{-1}(\log\lvert x\rvert - c)$
10.	Show that the following differential equation is homogeneous and then solve it

	$ydx + x\log\left\|\frac{y}{x}\right\| dy - 2xdx = 0$ Ans:$\log\left\|\frac{\log\left(\frac{y}{x}\right)-1}{y} = C\right\|$
11.	Find the general solution of the differential equation $(x+y)\frac{dy}{dx} = 1.$ x+ y +1 = ce^{y}
12.	Find the general solution of the differential *equation* $xlogx\ \frac{dy}{dx} + y = \frac{2}{x}\log x$ y log x = -$\frac{2}{x}$(1+log x) +c
13.	Solve the differential equation $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0, given\ y(0) = 0$ y(1+x^2) = $\frac{4}{3}$ x^3
14.	Write the integrating factor of differential equation $(1+x^2)\frac{dy}{dx} + (2xy - cotx) = 0$ y(1+x^2) = log $\|sinx\|$+ c
15.	Find the integrating factor of differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$ $ye^{2\sqrt{x}} = 2\sqrt{x} + c$

CREATIVE AND CRITICAL THINKING QUESTION

DIFFERENTIAL EQUATIONS

The most of the students of class XII of a school participate in Sports & Games and Co-Curricular activities.

The rate of increase of the students, who take part in Sports & Games, with respect to the change in the number of students, who take part in Co-Curricular activities is proportional to the ratio of the difference of the students of both the activities to the sum of the number of students of the activities.

Assume that ‘x’ students take part in Sports & Games activities and ‘y’ students take part in Co-Curricular activities and $x > y$.

Initial conditions: (1) Given $x = 1; y = 0$.

(2) Also given that $x = 3\ when\ y = 4$

Based on the above information answer the following questions:

(1) The relation between $\frac{dx}{dy}$ and x,y is

(i) $\frac{dy}{dx} \propto \frac{x-y}{x+y}$ (ii) $\frac{dx}{dy} \propto \frac{x-y}{x+y}$ (iii) $\frac{dx}{dy} \propto \left(\frac{x+y}{x-y}\right)$ (iv) none of these

(2) The Differential Equation corresponds to the above situation is

$(i)\ \frac{dy}{dx} = k\left(\frac{x-y}{x+y}\right)$ (ii) $\frac{dx}{dy} = k\left(\frac{x-y}{x+y}\right)$ (iii) $\frac{dx}{dy} = k\left(\frac{x+y}{x-y}\right)$ (iv) none of these.

(3) What is the substitution to be given to reduce the D.E. to Variable - Separable?

(i) $y = vx$ (ii) $x = vy$ (iii) $x = \lambda x\ and\ y = \lambda y$ (iv) none of these

(4) What is the general solution of this D.E?

(i) $\log\left|\sqrt{x^2+y^2}\right| = c + \tan^{-1}\left(\frac{x}{y}\right)$ (ii) $\log\left|\sqrt{x^2+y^2}\right| = c + \tan^{-1}\left(\frac{y}{x}\right)$

(ii) $\log|x^2+y^2| = c - \tan^{-1}\left(\frac{x}{y}\right)$ (iv) none of these.

(5) What is the particular solution of this D.E.?

(i) $\log\left|\sqrt{x^2+y^2}\right| = \log 5 + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{x}{y}\right)$

(ii) $\log\left|\sqrt{x^2+y^2}\right| = \log 5 + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

(iii) $\log\left|x^2+y^2\right| = \log 5 + \tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{x}{y}\right)$

(iv) None of these.

2. Money earns interest. The interest can be calculated at fixed times, such as yearly, monthly, etc. and added to the original amount. This is called <u>compound interest</u>.

But when it is compounded **continuously** then at any time the interest gets added in proportion to the current value of the loan (or investment).

And as the loan grows it earns more interest .assume the rate of change of current value of the loan w.r.t time is exactly equal to product of the rate of interest and current value

Using **t** for time, **r** for the interest rate and **V** for the current value of the loan:

Answer the following questions based on the above situations

1. Write the differential equation corresponding to the above situation

2. Mention the type of differential equation

3. Write the general solution of the differential equation

4. Find the current value of loan I of \$1,000 for 2 years at an interest rate of 10%

Solution

1. $\frac{dv}{dt} = rV$
2. variable separable type
3. $V = Pe^{rt}$
4. a continuously compounded loan of $1,000 for 2 years at an interest rate of 10% becomes:

$$V = 1000 \times e^{(2\times0.1)}$$

$V = 1000 \times 1.22140...$
$V = \$1,221.40$ (to the nearest cent)

1. Let P denote the principal deposited with a bank which gives an interest of r% per year.Ram invests

Rs 1000 in a bank which pays 5% interest per year .

Answer the following questions based on the above situations

1. The differential equationgoverning the growth of the principal is……..

2. What is the general solution?

3. The time required to double the principal is approximately…..[Use $\log_e 2 = .6931$]

a)11 years b) 12years c) 13 years d)14 years

4. What will be the amount in 10 years? [Use $e^{0.5}$=1.648]

a)Rs.1390 b)Rs,1648 c)Rs.2718 d)1448

Solution

1. $\frac{dp}{dt}$=KP K is constant 2. P=Ce^{kt}

3. 14 years approximately. 4. Rs, 1648

Error analysis in Differential equation

1. **Question: Find integrating factor of differential equation $x\frac{dy}{dx} + 2y = x^2$.**

Incorrect solution

P=2

Q=x^2

I F=$e^{\int Pdx}$ =$e^{\int 2dx} = e^{2x}$

Type of error : Concept error

Correct Answer

Divide equation by x

$\frac{dy}{dx} + \frac{2}{x}y = x$ Which is in the form $\frac{dy}{dx} + Py = Q$

So P = $\frac{2}{x}$

Q=x

I F=$e^{\int Pdx}$ =$e^{\int \frac{2}{x}dx} = e^{2logx} = e^{logx^2}$-=$x^2$

More problems for practice:

1. $x^2\frac{dy}{dx} - y = e^x$
2. $x^2\frac{dy}{dx} + y = e^x$
3. $x\frac{dy}{dx} + 2y = x^2$
4. $cosx\frac{dy}{dx} - ysinx = 1$
5. $(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$

2. $solve\ \frac{dy}{dx} = x^{-1} + 1$

dy = $(x^{-1}$ +1)dx

$$\int dy = \int (x^{-1} + 1)\ dx$$

wrong applications of integration rule

Y = $\frac{x^0}{0} + x + C$

Correct answer

Y= $\log|x|$ + x +c ($\int \frac{1}{x}\ dx = \log|x|$

2. **solve x dy +y dx =0**

wrongly done variables were not separated correctly

$\int$ x dy = - $\int$ y dx

$\frac{x^2}{2} = -\frac{y^2}{2}$ +C

Correct answer

$\int \frac{1}{y}$dy = - $\int \frac{1}{x}$ dx

logy = -log x +log C

Or xy =c

<u>Question:</u>

Solve the following homogeneous differential equation and find the particular solution:

$\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy = 0$. Given that $x = 0\ when\ y = 1$

WRONG STEPS

$$\frac{dx}{dy}=\frac{\left(1+e^{\frac{x}{y}}\right)}{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}$$

Error: In the RHS the coeff. of dx and dy are placed instead of dx and dy while skipping a few steps to save time.

(OR)

Sometimes the students may find the

$$\frac{dy}{dx}=\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{\left(1+e^{\frac{x}{y}}\right)}$$

$$\frac{dx}{dy}=\frac{e^{\frac{x}{y}}\left(\frac{x}{y}-1\right)}{\left(1+e^{\frac{x}{y}}\right)}$$

$$v+y\frac{dv}{dy}=\frac{e^{v}(v-1)}{1+e^{v}}$$

$$y\frac{dv}{dy}=\frac{e^{v}(v-1)}{1+e^{v}}-1$$

$$y\frac{dv}{dy}=\frac{ve^{v}-e^{v}-v-ve^{v}}{1+e^{v}}$$

$$y\frac{dv}{dy}=\frac{-(e^{v}+v)}{1+e^{v}}$$

$(1+e^{v})dv \quad dv$

CORRECT STEPS

$$\frac{dx}{dy}=\frac{e^{\frac{x}{y}}\left(\frac{x}{y}-1\right)}{\left(1+e^{\frac{x}{y}}\right)}$$

Transpose one of the terms to other side and then find $\frac{dy}{dx}$ or $\frac{dx}{dy}$

If the argument of any function is $\frac{y}{x}$ or $\frac{x}{y}$ then the d.e. is of homogeneous type.

For $\frac{y}{x}$, the substitution is $y=vx$; $\frac{dy}{dx}=v+x\frac{dv}{dx}$

and for $\frac{x}{y}$, the substitution is $x=vy$

$dx \quad dv$

$$\frac{dx}{dy}=\frac{e^{\frac{x}{y}}\left(\frac{x}{y}-1\right)}{\left(1+e^{\frac{x}{y}}\right)}$$

$$v+y\frac{dv}{dy}=\frac{e^{v}(v-1)}{1+e^{v}}$$

$$y\frac{dv}{dy}=\frac{e^{v}(v-1)}{1+e^{v}}-1$$

$$y\frac{dv}{dy}=\frac{ve^{v}-e^{v}-v-ve^{v}}{1+e^{v}}$$

$$y\frac{dv}{dy}=\frac{-(e^{v}+v)}{1+e^{v}}$$

$(1+e^{v})dv \quad dv$

$$\frac{(1+e^{v})dv}{v+e^{v}} = \frac{dy}{y}$$

$$\log|v+e^{v}| = -\log y + \log c$$

$$\log\left|x + ye^{\frac{x}{y}}\right| = \log c$$

$$x + ye^{\frac{x}{y}} = c$$

Forgetting to find the particular solution.

$$\frac{(1+e^{v})dv}{v+e^{v}} = \frac{dy}{y}$$

$$\log|v+e^{v}| = -\log y + \log c$$

$$\log\left|x + ye^{\frac{x}{y}}\right| = \log c$$

$$x + ye^{\frac{x}{y}} = c$$

Which is the **general solution**.

For particular solution $x = 0 \; when \; y = 1$

The particular solution is

$$x + ye^{\frac{x}{y}} = 1$$

Misconceptions :-

Q.N 1:- Solve the differential equation : $\frac{dy}{dx}$ = (1 +x^2)(1 + y^2)

Wrong Solution : Given D.E. is

$$\frac{dy}{dx} = (1+x^2)(1+y^2)$$

$$(1+y^2)\,dy = (1+x^2)dx \quad (\text{ transposition mistake})$$

On integration $\int(1 + y^2)\,dy = \int(1 + x^2)\,dx$

$$y + \frac{y^3}{3} = x + \frac{x^3}{3} + C \qquad \text{Ans.}$$

Correct Solution: - Given D.E. is

$$\frac{dy}{dx} = (1+x^2)(1+y^2)$$

$$\frac{1}{(1+y^2)}dy = (1+x^2)dx$$

On integration $\int \frac{1}{(1+y^2)}dy = \int(1 + x^2)\,dx$

$$\tan^{-1} y = x + \frac{x^3}{3} + C$$

Q.N 2: Find the general solution of the D.E

$$\left(x\, sin^2\left(\frac{y}{x}\right) - y\right) dx + xdy = 0$$

Wrong Solution : Given D.E.

$$(x\, sin^2\left(\frac{y}{x}\right) - y)\ dx + xdy = 0$$

$\frac{dy}{dx} = \frac{y}{x} + sin^2\left(\frac{y}{x}\right)$, which is homogeneous (sign mistake)

Let $\frac{y}{x} = v \Longrightarrow y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now $\quad v + x\frac{dv}{dx} = v + sin^2(v)$

$$x\frac{dv}{dx} = sin^2(v)$$

$$\frac{1}{sin^2 v}\, dv = \frac{1}{x}\, dx$$

$\log sin^2 v = \log x + C$ (wrong formula)

$\log sin^2\left(\frac{y}{x}\right) = \log x + C$ Ans.

Correct Solution : Given D.E.

$$(x\, sin^2\left(\frac{y}{x}\right) - y)\ dx + xdy = 0$$

$\frac{dy}{dx} = \frac{y}{x} - sin^2\left(\frac{y}{x}\right)$, which is homogeneous

Let $\frac{y}{x} = v \Longrightarrow y = vx \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now $\quad v + x\frac{dv}{dx} = v - sin^2(v)$

$$x\frac{dv}{dx} = - sin^2(v)$$

$$\frac{1}{sin^2 v}\, dv = -\frac{1}{x}\, dx$$

$$- \cot v = - \log x - C$$

$$\cot\left(\frac{y}{x}\right) = \log x + C$$

Ans.

Q.N 3:- Solve the differential equation : $x.\frac{dy}{dx} + y = x$

Wrong Solution :- Given D.E. is

$$x.\frac{dy}{dx} + y = x$$

I.F. = $e^{\int 1dx}$ = e^x (Conceptual mistake)

Solution is

$$ye^x = \int xe^x\ dx + C = x.e^x - \int e^x\, dx + C$$

$ye^x = x.e^x - e^x + C$ Ans.

Correct Solution :- Given D.E. is

$$x.\frac{dy}{dx} + y = x$$

$$\frac{dy}{dx} + \frac{1}{x}y = 1$$

I.F. = $e^{\int \frac{1}{x}dx}$ = $e^{\log x}$ = x

Solution is

$$yx = \int x\, dx + C = \frac{x^2}{2} + C$$

$yx = \frac{x^2}{2} + C$ Ans.

Q.N 4:- Solve the differential equation : $\frac{dy}{dx} = e^{x+y}$

Wrong Solution : Given D.E. is $\frac{dy}{dx} = e^{x+y}$

$$dy = e^{x+y}\, dx$$

on integrating, $\int dy = \int e^{x+y} dx$

$y = e^{x+y} + C$ (Misconception) Ans.

Correct Solution : Given D.E. is $\frac{dy}{dx} = e^{x+y} = e^x.e^y$

$$e^{-y}\, dy = e^x\, dx$$

on integrating, $e^{-y} = e^{x}+C$

$y = e^{x+y} + C$ Ans.

<u>Non Routine Questions on D.E.:-</u>

Q.N. 1:- Solve the differential equation: $\frac{d^2y}{dx^2} = \cos x$

Solution:- Given D.E.

$$\frac{d^2y}{dx^2} = \cos x$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \cos x$$

$$\int \frac{d}{dx}\left(\frac{dy}{dx}\right) dx = \int \cos x \, dx$$

$$\frac{dy}{dx} = \sin x + C_1$$

On integrating again

$$\int \frac{dy}{dx} dx = \int (\sin x + C_1) \, dx$$

$$y = -\cos x + C_1x + C_2 \quad \text{Ans.}$$

Q.N.: Solve the differential equation: $\frac{dy}{dx} = (x+y)^2$

Solution: Given D.E

$$\frac{dy}{dx} = (x+y)^2 \quad \ldots\ldots\ldots\ldots(1)$$

Let $x + y = v \implies 1 + \frac{dy}{dx} = \frac{dv}{dx}$

Now from (1) $\frac{dv}{dx} - 1 = v^2$

$$\frac{1}{1+v^2} dv = dx$$

On integrating

$$\int \frac{1}{1+v^2} \, dv = \int 1 \, dx$$

$\tan^{-1} v$ = x + C

$\tan^{-1}(x + y)$ = x + C Ans

S.no.	topic	Errors committed by the students	remedial
1	Differential equations	Writing incorrect order of differential equations	order is the order of highest order derivative
2	Differential equations	Writing incorrect degree of differential equation	Write degree of highest derivative after removing all radical signs
3	Differential equations	Wrong identification of form of Differential equations	Form of Differential equations like variable separable ,homogeneous and linear should be explained in comparative manner with three examples.
4	Differential equations	Incorrect integration of the given expression	Basics of integration should be discussed
5	Differential equations	Constant of integration	It should be used with every solution

CONCEPT MAPPING OF VECTOR

Definition of a vector → Vector is a physical quantity having magnitude and direction, a directed line segment is used to represent it, length of the line segment gives magnitude and arrow mark gives direction.

$\overrightarrow{AB}$, A is the Initial point and B is the terminal point.

Example–velocity, acceleration, force, momentum etc

1. **Type of vectors** →

Zero vector → magnitude zero and arbitrary direction or initial and terminal points coincides in a vector.

a) Unit vector → magnitude one unit and having a definite direction or particular direction.
b) Equal vectors → having equal magnitude and same direction
c) Like vectors→ having same direction.
   Unlike vectors → having opposite direction.
d) Co - initial vectors → having same initial point.
   Collinear vectors → which can be represented on a same line.
e) Coplanar vectors → vectors lying on the same plane.
f) Negative of a vector → a vector having same magnitude and opposite direction as that of a given vector.
g) Position vector of a point → position vector of a point P is $\overrightarrow{OP}$ where O is the origin.
h) Any vector $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$.
i) If P is any point (x, y, z) then $\overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

3) **Operation on vectors**; addition of vectors → having same direction, opposite directions and different directions.

   Triangle law of vector addition →if two vectors are represented by two sides of a triangle taken in order then their resultant is represented by the third side taken in the opposite direction.

Parallelogram law of vector addition → if two vectors are represented by two adjacent sides of a parallelogram, then their resultant is represented by the diagonal of the parallelogram passing through the common vertex of the adjacent sides.

4) **Properties of vector addition** → Commutative property → $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

Associative property → $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

Identity property → $\vec{a} + \vec{o} = \vec{0} + \vec{a}$.

Inverse property → $\vec{a} + (-\vec{a}) = \vec{o}$

5) **Subtraction of vectors.** $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

6) **Multiplication of a vector by a scalar** → if k is a scalar and $\vec{a}$ is a vector then $k\vec{a}$ is a vector whose magnitude is k times that of $\vec{a}$ and direction is same or opposite according as k is positive or negative.

Distributive property → $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$.

7) **Definition of dot (scalar) product of vectors—** $\vec{a} . \vec{b} = ab\cos\theta$.

Angle between two vectors is given by $\cos\theta = \frac{\vec{a} . \vec{b}}{ab}$.

Condition for perpendicularity of two vectors as $\vec{a} . \vec{b} = 0$.

Geometrical meaning → $\vec{a} . \vec{b} =$ (Magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$)

= (Magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$)

Orthonormal triads; $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ → $\hat{\imath} . \hat{\imath} = \hat{\jmath} . \hat{\jmath} = \hat{k} . \hat{k} = 1$ and $\hat{\imath} . \hat{\jmath} = \hat{\jmath} . \hat{k} = \hat{k} . \hat{\imath} = 0$.

POINTS TO REMEMBER.

1) $\vec{a} . \vec{b} = ab\cos\theta$.

2) $\cos\theta = \frac{\vec{a} . \vec{b}}{ab}$.

3) Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a} . \vec{b}}{|\vec{b}|}$.

4) If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ then $\vec{a} . \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

4) If $\vec{a}$ and $\vec{b}$ are perpendicular then $\vec{a}\,.\vec{b} = 0$.

5) $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\,.\vec{b})$

6) $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}\,.\vec{b})$.

7) $(\vec{a} + \vec{b}) \,.\, (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$.

8) If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ then $|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2$.

9) $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$.

10) $|\vec{a} + \vec{b} + \vec{c}\,|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a}\,.\vec{b} + 2\vec{b}\,.\vec{c} + 2\vec{c}\,.\vec{a}$.

11) If $\hat{a}$ is a unit vector and $\vec{a}$ is any vector then $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

TOPIC: CROSS PRODUCT OF VECTORS

I. CONCEPT MAPPING

1. Definition

2. Types

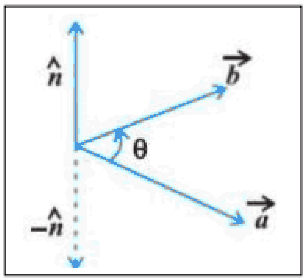
3. Formula

4. Concept of related terms

5. Problems

6. Example.

- **Vector product :** The vector product of two nonzero vectors $\vec{a}$ *and* $\vec{b}$, is denoted by $\vec{a} \times \vec{b}$ and defined as



$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n},$$

where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system (as shown in adjoining figure). i.e., the right handed system rotated from $\vec{a}$ *to* $\vec{b}$ moves in the direction of $\hat{n}$.

Observations

1. $\vec{a} \times \vec{b}$ is a vector.

2. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then $\vec{a} \times \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are parallel (or collinear) to each other, i.e.,

$$\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a} \,||\, \vec{b}$$

In particular, $\vec{a} \times \vec{a} = 0$ and $\vec{a} \times (-\vec{a}) = 0$, since in the first situation, θ = 0 and in the second one, θ = π, making the value of sin θ to be 0.

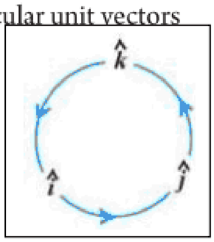
3. If $\theta = \frac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$.

4. In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{\imath}, \hat{\jmath}$ *and* $\hat{k}$, we have

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0},$

$\hat{\imath} \times \hat{\jmath} = \hat{k},\ \hat{\jmath} \times \hat{k} = \hat{\imath},\ \hat{k} \times \hat{\imath} = \hat{\jmath}$ (see Fig 7)

Fig. 7



5. In terms of vector product, the angle between two vectors $\vec{a}$ and $\vec{b}$ may be given as

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

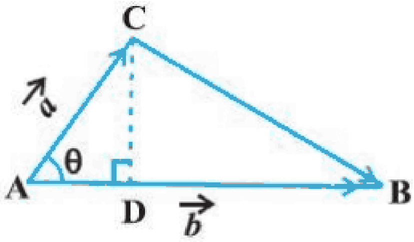
6. It is always true that the vector product is not commutative, as

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

7. In view of the Observations 4 and 6, we have

$$\hat{\jmath} \times \hat{\imath} = -\hat{k}, \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath}, \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

8. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given as $\frac{1}{2}|\vec{a} \times \vec{b}|$ = Area of triangle ABC



9. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a} \times \vec{b}|$

- **Two important properties of vector product** (Distributivity of vector product over addition): If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors and λ be a scalar, then
  - (i) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
  - (ii) $\lambda(\vec{a} \times \vec{b}) = (\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b})$
- If $\vec{a}$ *and* $\vec{b}$ are any two vectors given in the component form $a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ *and* $b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, respectively, then their cross product may be given by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

VECTORS (CROSS PRODUCT)

LEVEL 1

1.	Write the value of $(\hat{\imath} \times \hat{\jmath}).\hat{k} + (\hat{\jmath} \times \hat{k}).\hat{\imath}$
2.	Find the area of the parallelogram determined by the vectors $2\hat{\imath}$ *and* $3\hat{\jmath}$
3.	Find a unit vector perpendicular to both of the vectors $\underline{a} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ *and* $\underline{b} = -\hat{\jmath} - 2\hat{k}$
4.	Using vectors, find the area of the triangle with the vertices A (1,1,2), B(2,3,5) and C(1,5,5)
5.	If $\vec{a} + \vec{b} + \vec{c} = \underline{0}$, *then prove that* $\vec{a} \times \underline{\vec{b}} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

ANSWER : 1) 2

2) 6 sq. Units

3) $\frac{1}{2\sqrt{6}}(-2\hat{\imath} + 4\hat{\jmath} - 2\hat{k})$

4) $\frac{1}{2}\sqrt{61}$ sq. units

LEVEL 2

1.	Let $\underline{a}$ *and* $\underline{b}$ *be two vectors such that* $|a| = 3$ *and* $|b| = \frac{\sqrt{2}}{3}$ *and* $\underline{a} \times \underline{b}$ *is a unit vector,* *what is the angle between* $\underline{a}$ *and* $\underline{b}$?
2.	Find the value of p, If $(2\hat{\imath} + 6\hat{\jmath} + 27\hat{k}) \times (\hat{\imath} + 3\hat{\jmath} + p\hat{k}) = \underline{0}$
3.	If $\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, *find* $(\underline{r} \times \hat{\imath}).(\underline{r} \times \hat{\jmath}) + xy$
4.	If $\underline{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, $\underline{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\underline{c} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$, *verify that* $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$

5.	The two adjacent sides of a parallelogram are $2\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$ and $\hat{\imath} - 2\hat{\jmath} - 3\hat{k}$. Find the area of this parallelogram

ANSWER : 1) $\theta = 45^0$

2) P=$\frac{81}{6} = \frac{27}{2}$

3) 0   5) $11\sqrt{5}\ Sq.units$

Level 3

1.	Find the unit vector which is perpendicular to both of the vector $\underline{a}$ = $3\hat{\imath}$ +$\hat{\jmath}$ +4$\hat{k}$and $\underline{b}$ = $\hat{\imath}$- $\hat{\jmath}$+ $\hat{k}$
2.	Let $\underline{a}$ = $\hat{\imath}$ - $\hat{\jmath}$, $\underline{b}$ = 3$\hat{\jmath}$ -$\hat{k}$ and $\underline{c}$ = 7$\hat{\imath}$-$\hat{k}$. Find a vector **d** which is perpendicular to both $\underline{a}$ and $\underline{b}$ and $\underline{c}.\underline{d}$ =1
3.	If $\underline{a}$,$\underline{b}$,$\underline{c}$ are three vectors such that $\underline{a}.\ \underline{b}$ =$\underline{a}.\ \underline{c}$ and $\underline{a}$x$\underline{b}$= $\underline{a}$ x$\underline{c}$, $\underline{a}$ ≠0 then show that $\underline{b}$ = $\underline{c}$
4.	Find the sine of the angle between the vectors $\underline{a} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ and $\underline{b} = 2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$
5.	For vectors $\underline{a}, \underline{b}$ and $\underline{c}$, prove that $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = 0$

ANSWER : 1) $\frac{1}{\sqrt{42}}$

2) $\frac{1}{4}(\hat{\imath} + \hat{\jmath} + 3\hat{k})$

4) $\frac{2}{\sqrt{7}}$

1) Find the projection of $\hat{\imath}$ - $\hat{\jmath}$ on $\hat{\imath}$ + $\hat{\jmath}$.

2) If $|\vec{a}|$ =2, $|\vec{b}|$ =$\sqrt{3}$ and $\vec{a}.\vec{b}$ =$\sqrt{3}$.Find the angle between $\vec{a}$ and $\vec{b}$.

3) Find the value of λ when the projection of $\vec{a}$ =λ$\hat{\imath}$ + $\hat{\jmath}$ +4$\hat{k}$ on $\vec{b}$ =2$\hat{\imath}$ +6$\hat{\jmath}$+3$\hat{k}$ is 4 units.

4) If $\vec{a}.\vec{a}$ =0 and $\vec{a}.\vec{b} = 0$ Then what can be concluded about the vector $\vec{b}$?

5) Show that angle in a semicircle is a right angle using vectors.

(4 Marks)

6) Show that diagonals of a rhombus are perpendicular to each other using vectors.

7) Show that the vectors $\frac{1}{7}$ (2$\hat{\imath}$ +3$\hat{\jmath}$+6$\hat{k}$), $\frac{1}{7}$ (3$\hat{\imath}$ - 6$\hat{\jmath}$+2$\hat{k}$) and $\frac{1}{7}$ (6$\hat{\imath}$ +2$\hat{\jmath}$-3$\hat{k}$) are mutually perpendicular unit vectors.

8) Find the angle between the vectors $\hat{\imath}$ -2$\hat{\jmath}$+3$\hat{k}$ and 3$\hat{\imath}$ -2$\hat{\jmath}$+$\hat{k}$.

9) Find $|\vec{a} - \vec{b}|$ if $|\vec{a}|$ =2, $|\vec{b}|$ =3 and $\vec{a}.\vec{b} = 4$.

10) If $\vec{a}$ is a unit vector and ($\vec{x}$ - $\vec{a}$) . ($\vec{x}$ + $\vec{a}$) =15. Find $|\vec{x}|$.

11) If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} + \vec{b}|$ =$|\vec{a}|$ then prove that the vector $2\vec{a} + \vec{b}$ is perpendicular to $\vec{b}$.

12) Show that the angle between the diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

13) If vectors $\vec{a}$ =2$\hat{\imath}$ +2$\hat{\jmath}$+3$\hat{k}$, $\vec{b}$ =$\hat{\imath}$ +$\hat{\jmath}$+$\hat{k}$ and $\vec{c}$ =3$\hat{\imath}$ +$\hat{\jmath}$ are such that $\vec{a}$ +λ $\vec{b}$ is perpendicular to $\vec{c}$.Find the value of λ.

14) If $\hat{a}$ and $\hat{b}$ are two unit vectors and θ is the angle between them ,then prove that

a. sin θ /2 =$\frac{1}{2}$ $|\vec{a} - \vec{b}|$.

15) If $\hat{a}$ and $\hat{b}$ are unit vectors and θ is the angle between them ,then prove that tan θ/2 =$\left\lfloor \frac{\vec{a}-\vec{b}}{\vec{a}+\vec{b}} \right\rfloor$.

16) Find the value of p so that $\vec{a}$ =2$\hat{\imath}$ + p$\hat{\jmath}$ +$\hat{k}$ and $\vec{b}$ =$\hat{\imath}$ -2$\hat{\jmath}$+3$\hat{k}$ are perpendicular to each other

17) Find $|\vec{a}|$ if $|\vec{a}|$=2$|\vec{b}|$ and ($\vec{a}$ +$\vec{b}$) . ($\vec{a}$ -$\vec{b}$) = 12.

18) If $|\vec{a} + \vec{b}|$ =60, $|\vec{a} - \vec{b}|$ =40 and $|\vec{b}|$ =46. Find $|\vec{a}|$.

19) **IF $\vec{\boldsymbol{a}}$**=(3i+2j-3k) and $\vec{b}$ =4i+7j-3k Find vector projection of $\vec{\boldsymbol{a}}$ in the direction of $\vec{b}$.

20) The two adjacent sides of a parallelogram are $(2\hat{i}-4\hat{j}+5\hat{k})\,\&\,(\hat{i}-2j-3\hat{k})$. Find the unit vectors parallel to its diagonals. Also find its area.

21) If $(\hat{i}+\hat{j}+\hat{k}),(2\hat{i}+5\hat{j}-3\hat{k}),(3\hat{i}+2\hat{j}-2\hat{k})\,\&\,(\hat{i}-6j-\hat{k})$ are the position vectors of points A, B, C & D respectively, then find the angle between AB & CD. Deduce that AB & CD are parallel.

CONCEPT MAPPING OF THREE –DIMENSIONAL GEOMETRY

1. **Distance formula**: Distance between two points A(x_1 ,y_1, z_1) and B(x_2, y_2,z_2) is
   AB=$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
2. **Section formula**: Coordinates of a point P, which divides the join of two given points
   A(x_1 ,y_1, z_1) and B(x_2, y_2,z_2) in the ratio l:m
   (i). internally,
   are P($\left(\frac{lx2+mx1}{l+m}, \frac{ly2+my1}{l+m}, \frac{lz2+mz1}{l+m}\right)$,
   the Coordinates of a point Q dividing the join in the ratio l:m
   (ii). externally are Q$\left(\frac{lx2-mx1}{l-m}, \frac{ly2-my1}{l-m}, \frac{lz2-mz1}{l-m}\right)$
   (iii).coordinate of mid-point are R($\left(\frac{x1+x2}{2}, \frac{y1+y2}{2}, \frac{z1+z2}{2}\right)$
3. **Direction cosines of a line** :
   (i).The direction of a line OP is determined by the angles α, β, γ which makes with OX, OY,OZ respectively. These angles are called the direction angles and their cosines are called the direction cosines.
   (ii).Direction cosines of a line are denoted by l,m,n . l=cos α ,m=cos $\beta, n = \cos\gamma$
   (iii).Sum of the squares of direction cosines of a line is always 1.
   $l^2+m^2+n^2=1$ i.e $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

4. **Direction ratio of a line** :(i)Numbers proportional to the direction cosines of a line are called direction ratios of a line .If a ,b ,c, are , direction ratios of a line, then

$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

(ii). If a ,b ,c, are , direction ratios of a line , then the direction cosines are

$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}} \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

(iii). Direction ratio of a line AB passing through the points A(x_1,y_1,z_1)and B (x_2,y_2,z_2) are
$x_2 - x_1$, y_2-y_1, z_2-z_1

4. STRAIGHT LINE:. (i). Vector equation of a Line passing through a point $\vec{a}$ and along the direction $\vec{\boldsymbol{b}}$, : $\vec{r}=\vec{a}+\boldsymbol{\mu}\vec{\boldsymbol{b}}$,

(ii).Cartesian equation of a Line: $\frac{x-x1}{a}=\frac{y-y1}{b}=\frac{z-z1}{c}$. Where ($x_1, y_1, z_1$ is the passing through a point and along the direction ratios are a,b,c

6. (i). Vector equation of a Line passing through two points, with position vectors $\vec{a}$ *and* $\vec{b}$ $\vec{r}=\vec{a}+\boldsymbol{\mu}(\vec{\boldsymbol{b}}-\vec{a})$

(ii).).Cartesian equation of a Line: $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$, two points are ($x_1, y_1$) and ($x_2, y_2$).

7. **ANGLE** between two lines (i). Vector equations: $\vec{r}=\overrightarrow{a_1}+\boldsymbol{\lambda}\overrightarrow{b_1}$ and: $\vec{r}=\overrightarrow{a_2}+\boldsymbol{\mu}\overrightarrow{b_2}$,

$$\cos\theta=\frac{\overrightarrow{b1}.\overrightarrow{b2}}{|\overrightarrow{b1}|.|\overrightarrow{b2}|}$$

(ii)).Cartesian equations: $\frac{x-x1}{a1}=\frac{y-y1}{b1}=\frac{z-z1}{c1}, \frac{x-x2}{a2}=\frac{y-y2}{b2}=\frac{z-z2}{c2}$

$$\cos\theta=\frac{a1.a2+b1.b2+c1.c2}{\sqrt{a^2+b^2+c^2}\,\sqrt{a^2+b^2+c^2}}$$

(iii). If two lines are perpendicular, then $\overrightarrow{b1}.\overrightarrow{b2}=0$, $a_1a_2+b_1b_2+c_1c_2=0$

(iv) . If two lines are parallel, then $\overrightarrow{b1} = t\,\overrightarrow{b2}$, where t is a scalar. OR $\overrightarrow{b1}\times\overrightarrow{b2}=0$, $\frac{a1}{a2}=\frac{b1}{b2}=\frac{c1}{c2}$

(v).If θ *is the* angle between two lines with direction cosines ,l_1, m_1, n_1 and l_2, m_2, n_2 then

(a).$\cos\theta=l_1l_2+m_1m_2+n_1n_2$ (b). if the lines are parallel, then $\frac{l1}{l2}=\frac{m1}{m2}=\frac{n1}{n2}$

(c). if the lines are perpendicular, then $l_1l_2+m_1m_2+n_1n_2=0$

8.(a).Shortest distance between two skew- lines:

(i).Vector equations: $\vec{r}=\overrightarrow{a_1}+\boldsymbol{\lambda}\overrightarrow{b_1}$, and: : $\vec{r}=\overrightarrow{a_2}+\boldsymbol{\mu}\overrightarrow{b_2}$,

$$d=\left|\frac{(\overrightarrow{a2}-\overrightarrow{a1}).(\overrightarrow{b1}\times\overrightarrow{b2})}{|\overrightarrow{b1}\times\overrightarrow{b2}|}\right|.$$

If shortest distance is zero, then lines intersect and line intersects in space if they are coplanar. Hence if above lines are coplanar

If $(\overrightarrow{a2}-\overrightarrow{a1}).(\overrightarrow{b1}\times\overrightarrow{b2})=0$

(ii). Cartesian equations: $\frac{x-x1}{a1}=\frac{y-y1}{b1}=\frac{z-z1}{c1}, \frac{x-x2}{a2}=\frac{y-y2}{b2}=\frac{z-z2}{c2}$

$$D=\frac{\begin{vmatrix} x2-x1 & y2-y1 & z2-z1 \\ a1 & b1 & c1 \\ a2 & b2 & c2 \end{vmatrix}}{\sqrt{(b1c2-b2c1)^2+(c1a2-c2a1)^2+(a1b2-a2b1)^2}}$$

9.If shortest distance is zero, then lines intersect and line intersects in space if they are

coplanar. Hence if above lines are coplanar

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a1 & b1 & c1 \\ a2 & b2 & c2 \end{vmatrix}=0$$

(b).Shortest distance between two parallel lines: If two lines are parallel, then they are coplanar.

Let the lines be : $\vec{r}=\overrightarrow{a_1}+\boldsymbol{\lambda}\vec{b}$, and: : $\vec{r}=\overrightarrow{a_2}+\boldsymbol{\mu}\vec{b}$,

$$D=\left|\frac{\vec{b}\times(\overrightarrow{a2}-\overrightarrow{a1})}{|\vec{b}|}\right|$$

10.General equation of a plane in vector form :- It is given by $\vec{r}.\vec{n}+d=0$, $\vec{n}$ is a vector normal to plane.

11.General equation of a plane in Cartesian form :- $\boldsymbol{ax+by+cz+d=0}$ **,** Where a,b,c are direction ratios of normal to the plane.

12.General equation of a plane passing through a point :- if position vector of given point is $\vec{a}$ then equation is given by $(\vec{r}-\vec{a}).\vec{n}=0$, $\vec{n}$ is a vector perpendicular tothe plane.

13.General equation of a plane passing through a point :- if position vector of given point is $\vec{a}$ then equation is given by $(\vec{r}-\vec{a}).\vec{n}=0$, $\vec{n}$ is a vector perpendicular tothe plane.

14.*Gen*eral equation of a plane passing through a point :- if coordinates of point are(x, y, z) then equation is$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$, a,b,care direction ratios of a line perpendicular to the plane.

15.Intercept form of equation of a plane :-General equation of a plane which cuts off intercepts a, b and c on x-axis, y-axis, z-axis respectively is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

16. Equation of a plane in normal form:- $\vec{r}.\hat{n} =$ p,where $\hat{n}$ is a unit vector along perpendicular from origin and 'p' is distance of plane from origin.p is always positive.

17.Equation of a plane in normal form :- It is given by $lx + my + nz = p$, where l, m, n are direction cosines of perpendicular from origin and 'p' is distance of plane from origin. p is always positive.

18.Equation of a plane passing through three non-collinear points :- If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of three non-collinear points, then equation of a plane through three points is given by –

$$(\vec{r} - \vec{a}).\{(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\} = 0.$$

19. Equation of a plane passing through three non-collinear points(Cartesian system) :- If plane passing through points $(\boldsymbol{x_1}, \boldsymbol{y_1}, \boldsymbol{z_1})$, $(\boldsymbol{x_2}, \boldsymbol{y_2}, \boldsymbol{z_2})$ and $(\boldsymbol{x_3}, \boldsymbol{y_3}, \boldsymbol{z_3})$ then equation is-

$$\begin{vmatrix} (x - x_1) & (y - y_{1)} & (z - z_{1)} \\ (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \end{vmatrix} = 0$$

20.If $\boldsymbol{\theta}$ is angle between two planes $\vec{\boldsymbol{r}}.\overrightarrow{\boldsymbol{n_1}} + \boldsymbol{d_1} = \boldsymbol{0}$ and $\vec{\boldsymbol{r}}.\overrightarrow{\boldsymbol{n_2}} + \boldsymbol{d_2} = \boldsymbol{0}$ then

$\cos\theta = \frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}$

(i) If planes are perpendicular, then $\overrightarrow{n_1}.\overrightarrow{n_2}$ =0

(ii) If planes are parallel, then $\overrightarrow{n_1} \times \overrightarrow{n_2} = 0$ or $\overrightarrow{n_1} = t\overrightarrow{n_2}$,t is a scalar.

21. **If $\boldsymbol{\theta}$ is angle between two planes $\boldsymbol{a_1 x + b_1 y + c_1 z + d_1 = 0}$ *and* $\boldsymbol{a_2 x + b_2 y + c_2 z + d_2 = 0}$**

$$\textbf{Then } \cos\boldsymbol{\theta} = \frac{\boldsymbol{a_1 a_2 + b_1 b_2 + c_1 c_2}}{\sqrt{(\boldsymbol{a_1}^2 + \boldsymbol{b_1}^2 + \boldsymbol{c_1}^2)(\boldsymbol{a_2}^2 + \boldsymbol{b_2}^2 + \boldsymbol{c_2}^2)}}$$

(i) If planes are perpendicular ,then $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

(ii) If planes are parallel , then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

22. **If $\boldsymbol{\theta}$ is angle between line $\vec{r} = \vec{a} + \lambda\vec{m}$ and the plane $\vec{r}.\vec{n} + d = 0$,then $\sin\theta = \frac{\vec{m}.\vec{n}}{|\vec{m}|.|\vec{n}|}$**

(i)If line is parallel to plane ,then $\vec{m}.\vec{n}$ =0 and

(ii)If line is perpendicular to plane , then $\vec{m} \times \vec{n} = 0$ or $\vec{m} = t\vec{n}$,t is a scalar.

23. . **If $\boldsymbol{\theta}$ is angle between line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and the plane $ax + by + cz + d = 0$,then**

$$\sin\theta = \frac{aa_1+bb_1+cc_1}{\sqrt{(a_1^2+b_1^2+c_1^2)(a^2+b^2+c^2)}}$$

(i)If line is parallel to the plane ,then $aa_1 + bb_1 + cc_1 = 0$

(ii)If line is perpendicular to the plane, then $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$

24. General equation of a plane parallel to the plane $\vec{r}.\vec{n} + d = 0$ is $\vec{r}.\vec{n} + \lambda = 0$, where λ is a constant and can be calculated from given condition.

25. General equation of a plane parallel to the plane ax+by+cz+d=0 is ax+by+cz+ $\lambda = 0$, where λ is a constant and can be calculated from given condition.

26. General equation of a plane (vector form) passing through the line of the intersection of planes

$\vec{r}.\overrightarrow{n_1} + d_1 = 0$ and $\vec{r}.\overrightarrow{n_2} + \lambda d_2 = 0$ is $\vec{r}.(\vec{n}_1 + \lambda\vec{n}_2) + (d_1 + \lambda d_2) = 0$, where λ is a constant and can be calculated from given condition.

27. General equation of a plane(Cartesian form) passing through the line of the intersection of planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is($a_1x+b_1y+c_1z+d_1$)+ λ ($a_2x+b_2y+c_2z+d_2$)=0 , where λ is a constant and can be calculated from given condition.

28. Distance of a plane(vector form) $\vec{r}.\vec{n} + d = 0$, from a point with position vector $\vec{a}$, is$\left|\frac{\vec{a}.\vec{n}+d}{|\vec{n}|}\right|$.

29. Distance of a plane(Cartesian form) ax+by+cz+d=0, , from a point (x_1,y_1,z_1) is $\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$.

QUESTIONS ON VECTORS & 3-D.

LEVEL-1

22) (4 Marks)

23) 3 Find the Co ordinate of foot of the perpendicular from origin to the plane 3x+4y-5z=7

(4 Marks)

24) Q 1: Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$

(6 Marks)

25) Q 2: Find the equation of the plane passing through the point (-1,3, 2) and perpendicular to the planes x +2y +3z = 5 and3x +3y + z = 0.

(6 Marks)

26) Q 3: Find the co-ordinates of the point where the line through (3, 4, 1) and (5, 1, 6) crosses the xy-plane . (6 marks)

LEVEL-2

1. (4 Marks)
2. 7. Find the distance of the point $(-1,-5,-10)$ from the point of intersection of the line $\vec{r}=(2\hat{\imath}-\hat{\jmath}+2\hat{k})+\lambda(3\,\hat{\imath}+4\hat{\jmath}+2\hat{k}\,)$ and the plane $\vec{r}.\left(\hat{\imath}-\hat{\jmath}+\hat{k}\right)=$
5. $ANS\text{: }13\ units.$ (6 Marks)
3. 8. A line makes angles $\alpha,\beta,\gamma\ and\ \delta$ with the four diagonals of a cube.
4. Prove that:$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$.

   (6 Marks)
5. Find the distance between the point P$(6,5,9)$ and the plane determined by the points A$(3,-1,2)$,B$(5,2,4)$ and C$(-1,-1,6)$. ANS:$\frac{6}{\sqrt{34}}$units.

   (6 Marks)
6. 10. Find the equation of the plane which is perpendicular to the plane 5x+3y+6z+8=0 and which contains the line of intersection of the planes x+2y+3z-4=0 and 2x+y-z+5=0. (6 Marks)

   ANS:51x+15y-50z+173=0

LEVEL-3

i. **Ans.**:-

$\frac{5\sqrt{2}}{2}; \vec{r} = -2\hat{i} + \lambda(\hat{i} + \hat{k})$

2. Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8=0$ and which contains the line of intersection of the planes $x+2y+3z-4=0 \; and \; 2x+y-z+5=0$. (6 Marks)

i. **Ans.**:-

$51x+15y-50z+173=0$

3. 12.Find the equation of a plane which is at a distance of 7 units from the origin and which is normal to the vector $3\hat{i}+5\hat{j}-6\hat{k}$. **Ans.**:-

$\vec{r}.(3\hat{i}+5\hat{j}-6\hat{k})-7\sqrt{70}=0$ (6 Marks)

4. . Find the distance of the point $(1,-2,3)$ *from the* plane x-y+z=5 measured || to the line$\frac{x+1}{2} = \frac{y+3}{3} = \frac{z+1}{-6}$. ANS: other point $(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7})$, $distance = 1\ unit$

(6 Marks)

ERROR ANALYSIS

Vectors (Cross product)

Using vectors, find the area of the triangle with the vertices A (1,1,2), B(2,3,5) and C(1,5,5)

Wrong answer by student:

Area of the triangle = ½ $\left|\hat{\imath} + \hat{\jmath} + 2\,\hat{k}) \times \left(\hat{\imath} + 5\,\hat{\jmath} + 5\hat{k}\right)\right|$

Identification of error: Vertices of triangle are given, students took these as sides.

Correct answer with explanation:

$$\underline{AB} = (2-1)\hat{\imath} + (3-1)\hat{\jmath} + (5-2)\,\hat{k}$$

$$= \hat{\imath} + 2\hat{\jmath} + 3\,\hat{k}$$

$$\underline{AC} = (1-1)\hat{\imath} + (5-1)\hat{\jmath} + (5-2)\,\hat{k}$$

$$=0\,\hat{\imath} + 4\hat{\jmath} + 3\,\hat{k}$$

$$\underline{AB} \times \underline{AC} = -6\hat{\imath} - 3\hat{\jmath} + 4\,\hat{k}$$

$$\left|\underline{AB} \times \underline{AC}\right| = \sqrt{36+9+16} = \sqrt{61}$$

$$1/2\left|\underline{AB} \times \underline{AC}\right| = 1/2\sqrt{36+9+16}$$

$$= 1/2\sqrt{61}$$

Follow up exercise :

1. $If\, a, \underline{b}$ and $\underline{c}$ $represents\ the\ vectors$ $\overrightarrow{BC}, \overrightarrow{CA}$ and $\overrightarrow{AB}$ of triangle ABC, then show that $\vec{b} \times \vec{c} = \vec{c} \times \vec{a} = \vec{a} \times \vec{b}$ Hence deduce sine formula for a triangle.

2. show that the points A, B and C with position vectors $\vec{a}$ = $\widehat{3\imath} - 4\hat{\jmath} - 4\,\hat{k}, \vec{b} = \widehat{2\,\imath} - \hat{\jmath} + \hat{k}$, $\vec{c}$ = $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ form the vertices of a right angled triangle.

3. Show that P (-2,4,7), Q (3, -6, -8) and R (1, -2, -2) are collinear.

4. Show that $\overrightarrow{(a - b)} \times \overrightarrow{(a + b)}$ = $2(\vec{a} \times \vec{b})$

5. If $\vec{d}_1$& $\vec{d}_2$ are the diagonals of the parallelogram with adjacent sides $\vec{a}$ *and* $\overrightarrow{b},$ Prove that area of the parallelogram is ½ $\left|\underline{\vec{d}1} \times \underline{d2}\right|$

Common Errors

1. Condition for parallelism and perpendicularity of 2 vectors
    a. If $\vec{a} = k\,\vec{b}$ or $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ (parallel)
    b. $a_1b_1 + a_2b_2 + a_3b_3 = 0$ (perpendicular)

2. Applying correct identity.
3. Dot and cross product of $\hat{\imath}$,$\hat{\jmath}$ and $\hat{k}$.
4. Taking general vector $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$.

TOPIC: THREE DIMENSIONAL GEOMETRY

GROUP V

Level 1

1	If the lines $\frac{x-1}{-2}=\frac{y-4}{3p}=\frac{z-3}{4}$ and $\frac{x-2}{4p}=\frac{y-5}{2}=\frac{z-1}{-7}$ are perpendicular to each other, then find the value of p.
2	Find the Cartesian and vector equation of the line which passes through the point (-2.4,-5) and parallel to the line given by $\frac{x+3}{3}=\frac{y-4}{5}=\frac{8-z}{-6}$
3	Find the angle between the lines $\frac{x-2}{3}=\frac{y+1}{-2}=z=2$ and $\frac{x-1}{1}=\frac{2y+3}{3}=\frac{z+5}{2}$
4	Find the Cartesian equation of a line passing through the point A(2,-1,3) and B(4,2,1). Also reduce it to vector form
5	The Cartesian equation of the line are 6x-2=3y+1=2z-2. Find its direction ratios and also find vector equation of the line.
6	A line makes angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ with x-axis and y-axis respectively. Find the angle made by the line with z-axis ?
7	Find the angle between the lines $\vec{r}=2\vec{i}-5\vec{j}+\vec{k}+\alpha(3\vec{i}+2\vec{j}+6\vec{k})$ and $\vec{r}=7\vec{i}-6\vec{k}+\mu(\vec{i}+2\vec{j}+2\vec{k})$
8	If α,β,γ $are\ the\ angles\ that\ a\ line\ makes\ with\ x, y\ and\ z\ axis$ $respectively$, then find $sin^2\alpha+sin^2\beta+sin^2\gamma$
9	Ifα $,\beta,\gamma$ $are\ the\ angles\ that\ a\ line\ makes\ with\ x, y\ and\ z\ axis$ $respectively$, then prove that $\cos 2\alpha+\cos 2\beta+\cos 2\gamma$ = - 1.
10	Find the angle between the straight lines : $\frac{x-1}{2}=\frac{y+1}{5}=\frac{z+3}{4}$ and $\frac{1-x}{-1}=\frac{y+2}{2}=\frac{3-z}{3}$
11	Find the shortest distance between the lines whose equations are $\frac{x+1}{2}=\frac{y-1}{-1}=\frac{z}{1}$ and $\frac{x-2}{3}=\frac{y-1}{-1}=\frac{z+1}{2}$
12	Find the shortest distance between the lines whose equations are $\frac{x}{2}=\frac{y+1}{-1}=\frac{z-3}{-2}$ and $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z+1}{1}$
13	Find the shortest distance between the lines whose equations are $\frac{x+1}{-2}=\frac{y+1}{-1}=\frac{z-1}{-1}$ and $\frac{x-2}{-3}=\frac{y-1}{1}=\frac{z+1}{-2}$

14	Find the shortest distance between the lines whose equations are $\frac{x-2}{2} = \frac{y+2}{-1} = \frac{z+2}{2}$ and $\frac{x+2}{3} = \frac{y+1}{-1} = \frac{z}{2}$
15	Find the shortest distance between the lines whose equations are $\frac{x+2}{-2} = \frac{y-1}{1} = \frac{z-1}{-1}$ and $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z+1}{1}$

LEVEL 2

1	Find the image of the point (1,6,3) in the line $\frac{x}{1} = \frac{y-1`}{2} = \frac{z-2}{3}$
2	Show that the line $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4`}{3} = \frac{z-6}{5}$ intersect, also find their point of intersection.If the points A(-1,3,2) and B(-4,2,-2) and C(5,5,λ) are collinear find the value of λ
3	Find the vector equation of the line joining (1,2,3) and (-3,4,3) and show that it is perpendicular to the z- axis.
4	What are the direction cosines of a line which makes equal angles with the coordinate axes?
5	Write the cartesian equation of the line PQ passing through points P(2, 2, 1) and Q(5,1,2). Hence, find the y-coordinate of the point on the line PQ whose z-coordinate is 2.
6	Find the value of p , so that the lines are at right angles. L_1 :$\frac{1-x}{3} = \frac{7y+14}{p} = \frac{z-3}{2}$ and L_2 : $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$
7	Show that the lines x = ay + b and z = cy + d and $x = a^1y + b^1$ and $Z = c^1y + d^1$ are perpendicular to each other , if $a a^1 + c c^1 + 1 = 0$.
8	Find the value of p , so that the lines are perpendicular to each other. L_1 :$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{1}$ and L_2 : $\frac{7-7x}{3p} = \frac{5-y}{1} = \frac{11-z}{7}$
9	If two lines L_1 and L_2 have their direction as $\vec{a} = (\vec{\imath} - \vec{\jmath} + 7\vec{k})$ and $\vec{b} = (5\vec{\imath} - \vec{\jmath} + \alpha\vec{k})$. Find the value of α if $\vec{a} + \vec{b}$ *and* $\vec{a} - \vec{b}$ are perpendicular to each other.
10	If $l_1 m_1, n_1$ and $l_{2,}\ m_2\ , n_2$ are direction cosines of two mutually perpendicular lines , show that the direction cosinesof a line perpendicular to both of these lines are $(m_1n_2 - m_2n_1\ , n_1l_2 - l_1n_2\ , l_1m_2 - m_1l_2)$

11	Show that the lines are intersecting. $\vec{r} = (3\vec{\imath} + 2\vec{\jmath} - 4\vec{k}) + \alpha(\vec{\imath} + 2\vec{\jmath} + 2\vec{k})$ $\vec{r} = (5\vec{\imath} - 2\vec{\jmath}) + \alpha(3\vec{\imath} + 2\vec{\jmath} + 6\vec{k})$
12	Show that the lines are intersecting. $\vec{r} = (\vec{\imath} + \vec{\jmath} - \vec{k}) + \alpha(3\vec{\imath} - \vec{\jmath})$ $\vec{r} = (4\vec{\imath} - \vec{k}) + \alpha(2\vec{\imath} + 3\vec{k})$.
13	Find the distance between lines. $\vec{r} = (-4\vec{\imath} + 4\vec{\jmath} + \vec{k}) + \alpha(\overrightarrow{8i} + 12\vec{\jmath} + 12\vec{k})$ $\vec{r} = (-3\vec{\imath} - 8\vec{\jmath} - 3\vec{k}) + \beta(2\vec{\imath} + 3\vec{\jmath} + 3\vec{k})$
14	Find whether the lines are intersecting. $\vec{r} = (\vec{\imath} - 2\vec{\jmath} - \vec{k}) + \alpha(3\vec{\imath} - \vec{\jmath} + 2k)$ $\vec{r} = (4\vec{\imath} + j - 2\vec{k}) + \alpha(2\vec{\imath} + \vec{k})$.
15	Find the distance between lines. $\vec{r} = (-2\vec{\imath} + \vec{k}) + \alpha(\vec{\imath} + \vec{\jmath} + \vec{k})$ $\vec{r} = (-\vec{\imath} - 2\vec{\jmath} - 3\vec{k}) + \beta(2\vec{\imath} + 3\vec{\jmath} - 3\vec{k})$

Level 3

1	Find the equation of the line which intersects the lines $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4}$ and $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and passes through the point (1,1,1).
2	Find the equation of the line through the point (1,-1,1) and perpendicular to the line joining the points (4,3,2) , (1,-1,0) and (1,2,-1) , (2,1,1)
3	Find the position vector of the foot of perpendicular drawn from point P(1,8,4) to the line joining A(0,-1,3) and B(5,4,4) also find the length of the perpendicular.
4	Show that the lines $\frac{1-x}{2} = \frac{y-3}{4} = \frac{z}{-1}$ and $\frac{x-4}{3} = \frac{2y-2}{-4} = \frac{z-1}{1}$ are coplanar
5	Find the value of λ so that the lines $\frac{1-x}{3} = \frac{7\mu-14}{\lambda} = \frac{z-3}{2}$ and $\frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angle and also find whether the lines are intersecting or not.

6	Verify that $\frac{l1 + l2 + l3}{\sqrt{3}}$, $\frac{m_1 + m_2 + m_3}{\sqrt{3}}$ and $\frac{n_1 + n_2 + n_3}{\sqrt{3}}$ are direction cosines of a line equally inclined to three mutually perpendicular lines with direction cosines ($(l_1, m_1, n_1), (l_2, m_2, n_2), and (l_3, m_3, n_3)$
7	Find the angle between the lines whose direction cosines are given by the equations l + m +n =0 and $l^2 + m^2 - n^2 = 0$
8	A line makes angles $\alpha, \beta, \gamma,\ and\ \delta$ with the four diagonals of a cube. Prove that $cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta = \frac{4}{3}$
9	Find the direction cosines of two lines connected by the relations l -5m +3n =0 and $7l^2 + 5m^2 - 3n^2 = 0$
10	Prove that the angle between any two diagonals of a cube is $\cos^{-1}\frac{1}{3}$
11	Find the Shortest Distance between the lines $\vec{r} = (t+1)\vec{i} + (2-t)\vec{j} + (1+t)\vec{k}$ $\vec{r} = (2s+2)\vec{i} - (1-s)\vec{j} + (2s-1)\vec{k}$
12	Find the Shortest Distance between the lines $\vec{r} = (1-t)\vec{i} + (t-2)\vec{j} + (3-2t)\vec{k}$ $\vec{r} = (s+1)\vec{i} + (2s-1)\vec{j} - (2s+1)\vec{k}.$
13	Define the shortest distance between 2 lines. Find the shortest distance and the vector Equation of the line of Shortest distance Between the lines. $\vec{r} = (-4\vec{i} + 4\vec{j} + \vec{k}) + \alpha(\vec{i} + \vec{j} - \vec{k})$ $\vec{r} = (-3\vec{i} - 8\vec{j} - 3\vec{k}) + \beta(2\vec{i} + 3\vec{j} + 3\vec{k})$
14	Define the shortest distance between 2 lines. Find the shortest distance and the vector Equation of the line of Shortest distance Between the lines. $\vec{r} = (-4\vec{i} + 4\vec{j} + \vec{k}) + \alpha(\vec{i} + \vec{j} - \vec{k})$ $\vec{r} = (-3\vec{i} - 8\vec{j} - 3\vec{k}) + \beta(2\vec{i} + 3\vec{j} + 3\vec{k})$
15	Find the Shortest Distance between the lines $\vec{r} = (\vec{i} + 2\vec{j} + 3\vec{k}) + \alpha(\vec{i} - 3\vec{j} + 2\vec{k})$ $\vec{r} = (4+2t)\vec{i} + (5+3t)\vec{j} + (6+t)\vec{k}$

CRITICAL THINKING

1.Two cars Ford and Amaze are running at the speed more than the allowed speed on the roads represented by the following Equations.

$$\vec{r} = \alpha(\vec{\imath} + 2\vec{\jmath} - \vec{k})$$

$$\vec{r} = (3\vec{\imath} + 3\vec{\jmath}) + \beta(2\vec{\imath} + \vec{\jmath} + \vec{k}).$$

a) Find the shortest distance between the above lines.
b) Will These 2 cars meet at a point ? If So, Find the Point of Intersection.
c) Find the angle between the roads.
d) Identify the position vector of a point on any road.

2. The Equation of motion of a rocket is x=3t , y =-4t , Z = t , time t is given in seconds and distance in kilometres.

a) Find the Path Traced by the Rocket.

1) Straight line 2) Parabola 3) Circle 4) Ellipse.

b) Which of the following points lie on the Path traced by the Rocket.

1) (6,8,2) 2) (6,-8,-2) 3) (6,-8,2) d) (-6,-8,-2)

c) At what distance will the rocket be from the starting point (0,0,0) from the Starting point in 5 seconds.

1) $\sqrt{450}$ 2) $\sqrt{550}$ 3) $\sqrt{250}$ 4) $\sqrt{650}$

d) If the position of a rocket at a certain instant of time is (5,-8,10) , Then what will be Height of the rocket from the Ground. (Ground is considered as XY plane.

1) 10 km 2) 12 km 3) 20 km 4) 11 km.

ANSWERS

1) a) 0 b) yes, (1,2,-1) c) 60^0 d) (3,3,0) or (0,0,0)
2) a) 1 b) 3 c)4 d) 1

ERROR ANALYSIS

TOPIC: THREE DIMENSIONAL GEOMETRY

1. QUESTION ON THE TOPIC:

Find the direction ratios of a line.

$\frac{1-x}{3} = \frac{7y+14}{5} = \frac{z-3}{2}$

WRONG ANSWER BY THE STUDENTS:

- ✓ (3, 5, 2)
- ✓ (-3, 7/5, 2)
- ✓ (-3, 7, 2)

IDENTIFICATION OF ERROR &

CORRECT ANSWER WITH EXPLANATION

(-3, 5/7, 2)

FURTHER EXPLANATION OF ERROR:

$$\frac{-(x-1)}{3} = \frac{7(y+2)}{5} = \frac{z-3}{2}$$

$$\frac{x-1}{-3} = \frac{(y+2)}{5/7} = \frac{z-3}{2}$$

FOLLOW UP EXERCISE WITH ANSWERS

A) Find the direction ratios of the lines:

$\frac{x-1}{2} = \frac{1-y}{5} = \frac{z+3}{4}$ ans: (2, -5, 4)

B) $\frac{1-x}{-1} = \frac{y+2}{2} = \frac{3-z}{3}$ ans: (1, 2, -3)

$C)$ $\frac{3x-1}{2} = \frac{y+1}{5} = \frac{5z+3}{4}$ ans: (2/3, 5, 4/5)

$D)$ $\frac{7x-21}{2} = \frac{3y+6}{5} = \frac{5z-25}{4}$ ans: (2/7, 5/3, 4/5)

$E)$ $\frac{2-x}{-2} = \frac{4y+1}{5} = \frac{z-3}{4}$ ans: (2, 4/5, 4

TOPIC: THREE DIMENSIONAL GEOMETRY

QUESTION ON THE TOPIC:

Find the image of the point (2 ,-1,5) in the line $\frac{x-11}{10} = \frac{y+2`}{-4} = \frac{z+8}{-11}$

WRONG ANSWER BY THE STUDENTS:

- ✓ Wrong substitution of line passing through the point
- ✓ Computational error in finding the co ordinates equating to k
- ✓ Wrong method of finding direction ratios of line
- ✓ Wrong usage of perpendicularity condition
- ✓ Wrong value of k
- ✓ Wrong value of co ordinates of foot of perpendicular
- ✓ Wrong concept of applying midpoint formula to get the image.

IDENTIFICATION OF ERROR &

CORRECT ANSWER WITH EXPLANATION

- ✓ correct substitution of line passing through the point (2, -1, 5)
- ✓ Finding the co ordinates equating to k(10k + 11, -4k - 2, -11k -8)
- ✓ Finding direction ratios of line (10k + 9, -4k - 1, -11k -13)
- ✓ perpendicular condition $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- ✓ correct value of k k = 1
- ✓ co ordinates of foot of perpendicular (1, 2, 3)
- ✓ the image (0, 5, 1)

LINER PROGRAMMING

CONCEPT MAPPING

Basic Concepts And Formulas:-

1. Definition:- Linear programming (LP) is an optimization technique in which a linear function is optimised (i.e. minimized or maximized) subject to certain constraints which are in the form of linear inequalities or/and equations. The function to be optimised is called objective function.
2. Application of linear programming:- Linear programming (LP) is used in determining optimum combination of several variables subject to certain constraints or restrictions.
3. Formation of linear programming problem (LPP): The basic problem in the formulation of a linear programming problem is to set-up some mathematical model. This can be done by asking the following question:

(a) What are the unknowns (variables)?

(b) What is the objectives?

(c) What are the restrictions?

For this, let $x_1, x_2, x_3, \ldots\ldots x_n$ be the variables. Let the objective function to be optimized

(i.e. minimized or maximized) be given by z.

(i) $Z = C_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$, where $c_i x_i$(i=1,2,3,….n) are constraints.

(ii) Let there be mn constants and let b_i be a set of constants such that

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n (\le, =, \ge) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n (\le, =, \ge) b_2$$

$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots$$

$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots$$

$$A_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n (\le, =, \ge) b_{1m}$$

(iii) Finally, let $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, \ldots\ldots, x_n \ge 0$, called non –negative constraints.

The problem of determining the values of $x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n$ which makes Z, a minimum or maximum and which satisfies(ii) and (iii) is called the general linear programming problem.

4.General LPP:

(a) Decision variables: The variables $x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n$ whose values are to be decided, are called decision variables.

(b)Objective function: The linear function $Z=c_1x_1+c_2x_2+\ldots\ldots\ldots\ldots+c_nx_n$, which is to be optimized (i.e. minimized or maximized)is called the objective function or preference function of the general linear programming problem.

(c) Structural constraints: The inequalities given in (ii), are called the structural constraints of the general linear programming problem. The structural constraint are generally in the form of inequalities of $\geq$ $type\ or \leq$ type,but occasionally, a structural constraint may be in the form of an equation.

(d) Non-negative constraints: The set of inequalities(iii) is usually known as the set of non-negative constraints of the general LPP. These constraints imply that the variables $x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n$, Cannot take negative values.

(e)Feasible solution: Any solution of a general LPP which satisfies all the constraints, structural and non-negative, of the problem, is called a feasible solution to the general LPP.

(f)Optimum solution: Any feasible solution which optimizes(i.e. minimized or maximized) the objective function of the LPP is called Optimum solution.

5.Requirement for Mathematical formulation of LPP:-Before getting the mathematical form of a linear programming problem, it is important to recognize the problem which can be handled by linear programming problem. For the formulation of a linear programming problem, the problem must satisfy the following requirements:

(i) There must be an objective to minimise or maximize something. The objective must be capable of being clearly defined mathematically as a linear function.

(ii)The resources must be in economically quantifiable limited supply .This gives the constraints to LPP.

(iii) The constraints(restrictions) must be capable of being expressed in the form of linear equations or inequalities.

6.Solving Linear programming problem: To solve linear programming problem .Corner Point Method is adopted, Under this method following steps are performed:

Step 1. At first , feasible region is obtained by plotting the graph of given linear constraints and its corner points are obtained by solving the two equations of the lines intersecting at that point.

Step 2. The value of objective function z = ax + by is obtained for each corner point by putting its x and y- coordinate in place of x and y in Z= ax + by .Let M and m be largest and smallest value of Z respectively.

Case-I: If the feasible region is bounded , then M and m are the maximum and minimum values of Z.

Case-II: If the feasible region is unbounded, then we proceed as follows:

Step-3: The open half plane determined by ax + by $> M$ and ax + by $< m$ are obtained.

Case I:If there is no common point in the half plane determined by ax + by $> M$ and feasible region, then M is maximum value of Z otherwise Z has no maximum value.

Case II: If there is no common point in the half plane determined by ax + by $< m$ and feasible region , then m is minimum value of Z otherwise Z has no minimum value.

Above facts can be represented by arrow diagram as:-

Feasible region (having largest and smallest values M and m of Z=ax+by at corner point)

Bounded

(M is maximum and m is minimum Value of Z)

Unbounded

If no common point in the half plane determined by $ax + by > M$ and feasible region, then M is maximum value of Z otherwise Z has no maximum value.

If there is no common point in the half plane determined by $ax + by < m$ and feasible region , then m is minimum value of Z otherwise Z has no minimum value.

LINEAR PROGRAMMING PROBLEMS :-

(I)Manufacturing Problems (profit always maximize/manufacturing cost always minimize)

(ii)Diet Problems (Cost Always Minimize)

(iii)Transportation Problems(Transportation Cost Always Minimize)

Note:- The feasible region of transportation problem has 2 pair of parallel lines.

TOPIC: LINEAR PROGRAMMING PROBLEMS

LEVEL 1 QUESTIONS	
Q.1	Maximize Z = 3x + 4y subject to constraints: $x + y \leq 4, x \geq 0, y \geq 0$
Q.2	Maximize Z = 3x + 2y subject to constraints: $x + 2y \leq 10, 3x + y \leq 15, x \geq 0, y \geq 0$
Q.3	Minimize Z = 3x + 5y subject to constraints: $x + 3y \geq 3, 3x + y \geq 2, x \geq 0, y \geq 0$
Q.4	Maximize Z = 5x + 3y subject to constraints: $3x + 5y \leq 15, 5x + 2y \leq 10, x \geq 0, y \geq 0$
Q.5	Maximize Z = 3x + 4y subject to constraints: $x + 2y \leq 8, 3x + 2y \leq 12, x \geq 0, y \geq 0$
LEVEL 2 QUESTIONS	
Q.6	Maximize Z = x + 2y subject to constraints: $2x + y \geq 3, x + 2y \geq 6, x \geq 0, y \geq 0$
Q.7	Minimize and Maximize Z = x + 2y subject to constraints: $x + 2y \geq 100, 2x - y \leq 0, 2x + y \leq 200, x \geq 0, y \geq 0$
Q.8	Maximize Z = 5x + 10y subject to constraints: $x + 2y \leq 120, x + y \geq 60, x - 2y \geq 0, x \geq 0, y \geq 0$
Q.9	Maximize Z = 3x + 4y subject to constraints: $x + 2y \leq 8, 3x + 2y \leq 12, x \geq 0, y \geq 0$
Q.10	Maximize Z = 3x + 4y subject to constraints: $x + 2y \leq 8, 3x + 2y \leq 12, x \geq 0, y \geq 0$
	LEVEL 3 QUESTIONS
Q.11	Maximize Z = -x + 2y subject to constraints: $x \geq 3, x + y \geq 5, x + 2y \geq 6, y \geq 0$
Q.12	Maximize Z =x + 2y subject to constraints: $x - y \leq -1, -x + y \leq 0, x \geq 0, y \geq 0$
Q.13	Solve the following linear programming problem graphically: Maximize Z = 4x + y subject to constraints: $x + y \leq 50, 3x + y \leq 90, x \geq 0, y \geq 0$
Q.14	Solve the following linear programming problem graphically: Maximize Z = 200x + 500y subject to constraints: $x + 2y \geq 10, 3x + 4y \leq 24, x \geq 0, y \geq 0$
Q.15	Solve the following linear programming problem graphically: Maximize Z = 3x + 9y subject to constraints: $x + 3y \leq 60, x + y \geq 10, x \leq y, x \geq 0, y \geq 0$

CRITICAL THINKING

CHAPTER : LINEAR PROGRAMMING PROBLEMS

QUESTION: Deepa rides her car at 25 km/hr. She has to spend Rs. 2/km on diesel and if she rides it at a faster speed of 40 km/hr, the diesel cost increases to Rs. 5/km. She has Rs. 100/- to spend on diesel. Let she travel x kms with speed 25 km/hr and y kms with speed 40 km/hr. Based on the information answer the following questions.

1. What is the outcome of this question?
2. What are the requirements are required by linear relationships?
3. If Z = x + y be the Objective function and Maximum value of Z is 30, at what point does the maximum value occur?
4. What is the point of intersection of the two lines?
5. What are the corner points?

ANSWER

1. Maximize $Z = x + y$
2. $x/25 + y/40 = 1$
3. $(50/3, 40/3)$
4. $(50/3, 40/3)$
5. $(0,0)$, $(25, 0)$, $(50/3, 40/3)$, $(0, 20)$

MANUFACTURING PROBLEMS

Question 1: A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftsman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftsman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.

(ii) What number of rackets and bats must be made if the factory is to work at full capacity?

(ii) If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.

Ans:- (max. profit Rs.200 when factory makes 4 tennis rackets and 12 cricket bats)

Question 2: A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit, of Rs 17.50 per package on nuts and Rs. 7.00 per package on bolts. How many packages of each should be produced each day so as to maximize his profit, if he operates his machines for at the most 12 hours a day?

Ans:- (max. profit Rs.73.5 when 3 package nuts &3 package bolt are produce)

Question 3: A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximize his profit? Determine the maximum profit.

Question 4: A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs. 5 and that from a shade is Rs. 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximize his profit?

Ans:-(manufacturer should produce 4 lamps & 4 shades to get maximum profiy of Rs. 32.)

Question 5: A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours of assembling. The profit is Rs. 5 each for type A and Rs. 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximize the profit?

LEVEL-2

Question 5: A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours of assembling. The profit is Rs. 5 each for type A and Rs. 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximize the profit?

Question6: A toy company manufactures two types of dolls, A and B. Market tests and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs. 12 and Rs. 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximize the profit?

Question 7: A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid at least 270 kg of potash and at most 310 kg of chlorine.

If the grower wants to minimize the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

kg per bag		
	Brand P	Brand Q
Nitrogen	3	3.5
Phosphoric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

Question 7: A merchant plans to sell two types of personal computers − a desktop model and a portable model that will cost Rs. 25000 and Rs. 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs. 70 lakhs and if his profit on the desktop model is Rs. 4500 and on portable model is Rs. 5000. What is the advantage of computer in daily life? Ans:- (Maximum profit of Rs. 11,50,000 is obtained when he stocks 200 desktop & 50 portable computer.)

Question 8: There are two types of fertilizers F_1 and F_2. F_1consists of 10% nitrogen and 6% phosphoric acid and F_2consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1cost Rs. 6/kg and F_2 costs Rs. 5/kg, determine how much of each type of fertilizer should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost? Why natural fertilizers are better than chemical fertilizers?

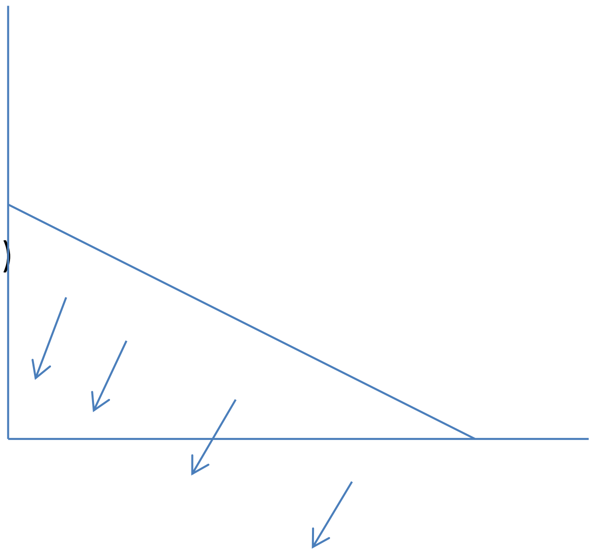
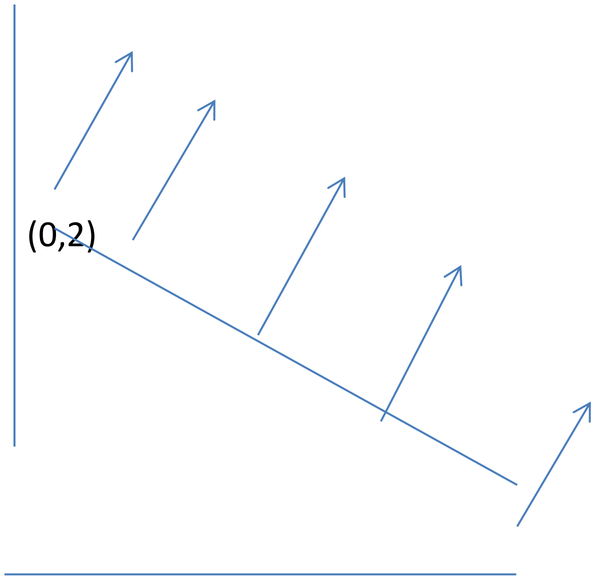
<u>COMMON MISTAKE(LPP)</u>

(1) **SOME CASES STUDENTS ARE NOT CLERIFY ABOUT EXACT SHADED REGION**.

ANS:- first we check the inequalities taking origine whether it satisfies or not , if it satisfies then the open half plane containing origin otherwise another half plane.

EXP-

$2X + 3y > 6$

Mistake graph	Correct graph
 (0,2) (3,0)	 (3,0)

(2) If at least & at most is comes then student doing mistake in symbols ($\leq$, $\geq$,

Ans:-if at least comes in the question then corresponding inequalities contain $\geq$, if at most is comes in the question then corresponding inequalities contain $\leq$.

EXP:- A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods $F_1$and F_2 are available. Food F_1 costs Rs 4 per unit food and

F_2 costs Rs 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem.

SOLUTION:-

Let the diet contain x units of food F_1 and y units of food F_2. Therefore,
$x \geq 0$ and $y \geq 0$
The given information can be complied in a table as follows.

	Vitamin A (units)	Mineral (units)	Cost per unit (Rs)
Food F_1 (x)	3	4	4
Food F_2 (y)	6	3	6
Requirement	80	100	

MISTAKES	CORRECTION
$3x + 6y \leq 80$	$3x + 6y \geq 80$
$4x + 3y \leq 100$	$4x + 3y \geq 100$

(3) When region is unbounded then some student do not clarify about max./min. Value is possible & not possible.

: The open half plane determined by ax + by $> M$ and ax + by $< m$ are obtained.

Case I:If there is no common point in the half plane determined by ax + by $>$ M and feasible region, then M is maximum value of Z otherwise Z has no maximum value.

Case II: If there is no common point in the half plane determined by ax + by <m and feasible region , then m is minimum value of Z otherwise Z has no minimum value.

<table>
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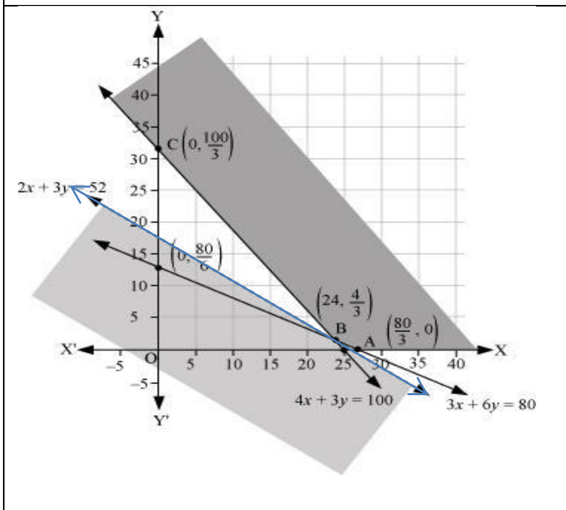
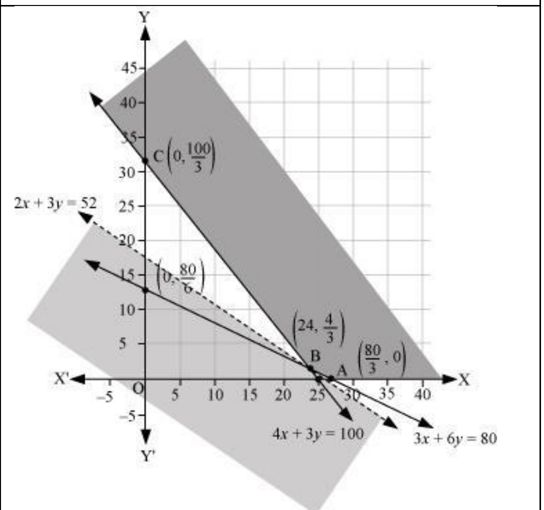
<tr><td>
</td><td>
</td></tr>
</table>

(4) STUDENTS ARE DOING MISTAKE NOT CONVERTING IN SAME UNIT.

EXP-Question 5:

A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximize his profit? Determine the maximum profit.

Answer :

Let the factory manufacture x screws of type A and y screws of type B on each day. Therefore,
$x \geq 0$ and $y \geq$
.

MISTAKE

	Screw A	**Screw B**	**Availability**
Automatic Machine (min)	4	6	4
Hand Operated Machine (min)	6	3	4

CORRECT

	Screw A	**Screw B**	**Availability**
Automatic Machine (min)	4	6	4×60 =240
Hand Operated Machine (min)	6	3	4×60 =240

PROBABILITY

CONCEPT MAPPING

1 .Definition of probability ,its formula .

2. Random Experiment ,outcomes ,sample space,events .

3. simple event ,compound events .

4.**Mutually exclusive events** : - Two events are said to be mutually exclusive if the occurrence of one prevents the occurrence of the other . in other words ,if A and B have no common elements , then they are said to be mutually exclusive events .

i.e. $A \cap B = \emptyset$.

5.**Conditional Probability** :- If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred,

i.e. P (E|F) is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \text{ ,provided } P(F) \neq 0 .$$

6. **Properties of conditional probability**

Let E and F be events of a sample space S of an experiment, then we have

Property 1: $P(S|F) = P(F|F) = 1$

Property 2: If A and B are any two events of a sample space S and F is an event

of S such that $P(F) \neq 0$, then $P((A \cup B)|F) = P(A|F) + P(B|F) – P((A \cap B)|F)$

Property 3: $P(E'|F) = 1 − P(E|F)$

7. **Multiplication Theorem on Probability**

$P(E \cap F) = P(E) P(F|E) = P(F) P(E|F)$, provided $P(E) \neq 0$ and $P(F) \neq 0$.

The above result is known as the multiplication rule of probability.

8. **Multiplication rule of probability for more than two events** :- If E, F and G are three events of sample space, we have $P(E \cap F \cap G) = P(E)\,P(F|E)\,P(G|(E \cap F)) = P(E)\,P(F|E)\,P(G|EF)$ Similarly, the multiplication rule of probability can be extended for four or more events.

9. **Independent Events** :- E and F are two events such that the probability of occurrence of one of them is not affected by occurrence of the other. Such events are called independent events .

OR

Let E and F be two events associated with the same random experiment,

then E and F are said to be independent if

$P(E \cap F) = P(E) . P(F)$ (it is known as multiplication theorem also)

Remarks

(i) Two events E and F are said to be dependent if they are not independent, i.e. if

$$P(E \cap F) \neq P(E) . P(F)$$

(ii) Mutually exclusive events never have an outcome common, but independent events, may have common outcome. Clearly, 'independent' and 'mutually exclusive' do not have the same meaning.

In other words, two independent events having nonzero probabilities of occurrence can not be mutually exclusive, and conversely, i.e. two mutually exclusive events having nonzero probabilities of occurrence can not be independent .

(iii) if the events E and F are independent, then

(a) E' and F are independent,

(b) E' and F' are independent

(c) If A and B are two independent events, then the probability of occurrence

of at least one of A and B is given by 1– P(A') P(B') .

10.**Addition Theorem** :- (a) When the events are not mutually exclusive :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(b) If Aand B are mutually exclusive i.e. $(A \cap B) = \emptyset$,then $P(A \cup B) = P(A) + P(B)$.

11. **Partition of a sample space** :- A set of events E1, E2, ..., En is said to represent a partition of the sample space S if

(a) $E_i \cap E_j = \phi$, $i \neq j$, i, j = 1, 2, 3, ..., n

(b) $E_1 \cup E_2 \cup \ldots \cup E_n = S$ and

(c) $P(E_i) > 0$ for all i = 1, 2, ..., n

.In other words, the events E1, E2, ..., En represent a partition of the sample spaceS if they are pairwise disjoint, exhaustive and have nonzero probabilities.

12. **Theorem of total probability**

Let {E1, E2,...,En} be a partition of the sample space S, and suppose that each of the

events E1, E2,..., En has nonzero probability of occurrence. Let A be any event associatedwith S, then

$$P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + \ldots + P(E_n)\, P(A|E_n) .$$

13. **Bayes' Theorem** :- If E1, E2 ,..., En are n non empty events which constitute a partitionof sample space S, i.e. E1, E2 ,..., En are pairwise disjoint and $E_1 \cup E_2 \cup \ldots \cup E_n = S$ andA is any event of nonzero probability, then

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n} P(E_i).P\left(\frac{A}{E_i}\right)} \quad \text{, for any i = 1, 2, 3, ..., n.}$$

14. Random Variable :- A random variable is a real valued function whose domain is the sample

space of a random experiment.

15. Probability Distribution : - The probability distribution of a random variable X is the system of numbers

X	x_1	x_2	$\boldsymbol{x_3}$	....	x_i	....	x_n
P(X)	p_1	p_2	p_3	....	p_i	....	p_n

where, p_i > 0 , $\sum p_i = 1$, i = 1, 2,..., n .

16. Mean of a random variable: -Let X be a random variable whose possible values x1, x2, x3, ..., xn occur with probabilities p1, p2, p3,..., pn, respectively. The mean of X, denoted by μ,

$$\mu = E(X) = \sum_{i=1}^{n} x_i p_i$$

And variance denoted by $\sigma^2 = \sum_{i=1}^{n} {x_i}^2 \, p_i - \mu^2$

Standard deviation = $\sqrt{variance}$

TOPIC : PROBABILITY

Level -1

1 If P(A) = 0.3, P(B) = 0.2, find P(B/A) if A and B are mutually exclusive events.

Given P(A) = $\frac{1}{4}$, P(B) = $\frac{2}{3}$ and P($A \cup B$) = $\frac{3}{4}$. Are the events independent ?

2 Find the probability of drawing two white balls in succession from a bag containing 3 red and 5 white balls respectively, the ball first drawn is not replaced.

3 'A' speaks truth in 60% cases and 'B' in 90% cases. In what % of cases are they likely to contradict each other in stating the same fact?

4 A problem in Mathematics is given to three students whose chances of solving it are $\frac{1}{3}$,$\frac{1}{6}$
$\frac{1}{5}$. What is the probability that at least one of them solves the problem.

5 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. Fint the probability that actually there was head .

6 A man is known to speak the truth in 75% cases. He throws a die and reports that it is a five. Find the probability that it is actually a five .

7 Often it is taken that a truthful person commands, more respect in the society. A man is known to speak truth 3 out of 4 times. He throws a die and report that it is a 6. Find the probability that it is actually 6.

8 A bag contains 4 balls. Two balls are drawn at random without replacement and are found to be blue. What is the probability that all balls in the bag are blue?

9 A insurance company has insured 4000 doctors, 8000 teachers and 12000 businessmen. The chances of a doctor, teacher and businessman dying before the age of 58 is 0.01, 0.03 and 0.05, respectively. If one of the insured people dies before 58, find the probability that he is a doctor.

10 Find k for the following probability distribution.

X	1	2	3	4	5
P(X)	0.1	0.4	0.2	0.15	K

And hence find $P(X > 3)$ and $P(2 \leq X < 5)$

11 Find the probability distribution of number of tails in a toss of two coins and hence find the mean.

12 Find the expectation of getting an outcome in a toss of a die.

13 Two cards drawn successively with replacement from a well shuffled pack of 52 card. Find the probability distribution of number of red cards.

14 Four defective bulbs are mixed with 10 good ones. Let X be the number of defective bulbs when 2 bulbs are drawn at random. Find the probability distribution of X.

Level -2

1. A dice is thrown twice and sum of numbers appearing is observed to be 6. what is the conditional probability that the number 4 has appeared at least once.

2 The probability of A hitting a target is $\frac{3}{7}$ and that of B hitting is $\frac{1}{3}$. They both fire at the target. Find the probability that (i) at least one of them will hit the target, (ii) Only one of them will hit the target.

3 A bag contains 5 white, 7 red and 3 black balls. If three balls are drawn one by one without replacement, find what is the probability that none is red.

4 Two balls are drawn at random with replacement from a box containing 10 blacks and 8 red balls. Find the probability that (i) both balls are red (ii) first ball is black and second is red.

5 The probability of student A passing an examination is $\frac{3}{5}$ and of student B passing is $\frac{4}{5}$
. Assuming the two events : 'A passes', 'B passes', as independent find the probability of:

a. Both students passing the examination
b. Only A passing the examination
c. Only one of the two passing the examination
d. Neither of the two passing the examination

6. It is observed that 50% of mails are spam. There is a software that filters spam mail before reaching the inbox. It accuracy for detecting a spam mail is 99% and chances of tagging a non-spam mail as spam mail is 5%. If a certain mail is tagged as spam find the probability that it is not a spam mail.

7. A card is lost from a pack of 52 cards. From the remaining cards two are drawn randomly and found to be both clubs. Find the probability that the lost card is also a clubs.
8. In shop A, 30 tin pure ghee and 40 tin adulterated ghee are kept for sale while in shop B, 50 tin pure ghee and 60 tin adulterated ghee are there. One tin of ghee is purchased from one of the shops randomly and it is found to be adulterated. Find the probability that it is purchased from shop B.
9. There are three identical cards except that both the sides of the first card is coloured red, both sides of the second card is coloured blue and for the third card one side is coloured red and the other side is blue. One card is randomly selected among these three cards and put down, visible side of the card is red. What is the probability that the other side is blue?

10 Two integers are selected at random from integers 1 to 11. If the sum of integers chosen is even, find the probability that both numbers are odd

11 If $P(X) = \begin{cases} 2kX, & X = 1,2 \\ X(1-k), & X = 3,4 \\ 0, & otherwise \end{cases}$

Find (i) k (ii) $P(X \leq 2)$

12 20% of the bulbs produced by a machine are defective. Find the probability distribution of the number of defective bulbs in a sample of 4 bulbs chosen at random.

13 Four rotten oranges are accidentally mixed with 16 good ones. Three oranges are drawn at random from the mixed lot. Let X be the number of rotten oranges drawn. Find the probability distribution of X and hence find the expectation of X.

14 In a game, 3 coins are tossed. A person is paid Rs.5 if he gets all heads or all tails; and he is supposed to pay Rs.3 if he gets one head or two heads. Find the probability distribution of the winning amount.

15 A biased coin which is 70% favourable to heads was tossed thrice, find the mean number of tails.

Level -3

1. If P(A) = $\frac{3}{8}$, P(B) = $\frac{1}{2}$ and $P(A \cap B) = \frac{1}{2}$, find $P(\overline{A}/\overline{B})$ and $P(\overline{B}/\overline{A})$

2 Two integers are selected from integers 1 through 11. If the sum is even, find the probability that both the numbers are odd.

3 A and B throw a die alternately till one of them gets a "6" and wins the game. Find the respective probability of winning, if A starts the game

4 A speaks truth in 60% of the cases and B in 70% of the cases. In what percentage of cases they are likely to (i) contradict each other (ii) agree with each other, in stating the same fact?

5 A and B throw a pair of dice alternately, till one of them gets a total of 10 and wins the game. Find their respective probabilities of winning, if A starts first.

6 In a neighbourhood, 90% children were falling sick due to flu and 10% due to measles and no other disease. The probability of observing rashes for measles is 0.95 and for flu is 0.08. If a child develops rashes, find the child's probability of having flu.

7 Bag 1 contains 3 red and 4 black balls and bag 2 contains 4 red and 5 black balls. Two balls are transferred at random from bag 1 to bag 2 then a ball is drawn from bag 2. The ball so drawn is found to be red in color. Find the probability that the transferred balls were both black.

8 An unbiased dice is rolled and for each number on the dice a bag is chosen:

Numbers on the Dice	Bag choosen
1	Bag A
2 or 3	Bag B
4 or 5 or 6	Bag C

Bag A contains 3 white ball and 2 black ball, bag B contains 3 white ball and 4 black ball and bag C contains 4 white ball and 5 black ball. Dice is rolled and bag is chosen, if a white ball is chosen find the probability that it is chosen from bag B.

9 Three urns are there containing white and black balls; first urn has 3 white and 2 black balls, second urn has 2 white and 3 black balls and third urn has 4 white and 1 black balls. Without any biasing one urn is chosen from that one ball is chosen randomly which was white. What is probability that it came from the third urn?

10 Three persons A, B and C have applied for a job in a private company. The chance of their selections is in the ratio 1 : 2 : 4. The probabilities that A, B and C can introduce changes to improve the profits of the company are 0.8, 0.5 and 0.3, respectively. If the change does not take place, find the probability that it is due to the appointment of C.

11 Let X denotes the number of times 'a total of 9' appears in two throws of a pair of dice. Find the probability distribution of X.

12 Find the probability distribution of getting either an ace or red card in draw of a card from a pack of 52 cards.

13 The probabilities that A, B and C speaks truth are 0.3, 0.2 and 0.5. Find the expected number of people speaks truth on giving a statement.

14 For a biased die, the probabilities for different faces to turn up are

Face	1	2	3	4	5	6
Probability	0.2	0.22	0.11	0.25	0.05	0.17

The above die tossed twice, Find the probability distribution of number of sixes.

15.Two cards are drawn successively with replacement from a well shuffled deck of 52 cards Find the probability distribution of number ace or face cards drawn.

CREATIVE AND CRITICAL THINKING

TOPIC : PROBABILITY

1	 Two biased coins are tossed together, let X and Y be random variables of number of heads and tails respectively such that $P(X=0)=P(X=1)=p$ and $E(X^2)=E(X)$, find 1) Find the value of p. 2) $P(X=2)$ 3) Find E(X). 4) Find P(Y=2) 5) Find P(X-Y is odd number) .
2	 Two dice are tossed. $A=\{(x,y):x+y=11\}$

	B= $\{(x, y): x = 5\}$ Where (x, y) denotes a typical sample point. 1) Find whether the following two events A and B are independent? 2)Find $P(A \cap B)$ 3) Find $P(A/B)$
3	 A letter is known to have come either from TATANAGAR or from CALCUTTA on the envelope, just two consecutive letters TA are visible. 1) Find the probability that the letter came from TATANAGAR. 2) Find the probability that the letter came from CALCUTTA 3) If the above two letters are mixed with another letter come from ITANAGAR on the envelope, then what is the probability that two consecutive letters TA are visible

1	Three students A, B, C can solve the problem independently, the probabilities of solving the problem of each of them are 0.3, 0.2 and 0.5 respectively, then find the probability that problem can be solved.
	Expected errors Students will add all three probabilities (Misconcepts) 0.2+0.3+0.5=1 Student may multiply all probabilities ((Misconcepts) 0.3*0.2*0.5=0.03 Calculation errors while doing additions and multiplications 1-0.7*0.8*0.5=1-2.8=-1.8 Correct answer P(problem can be solved)=1--0.7*0.8*0.5=1-0.28=00.72
	Similar questions 1. Three students A, B, C can solve the problem independently, the probabilities of solving the problem of each of them are $\frac{1}{3}, \frac{1}{5} \& \frac{1}{10}$ respectively, then find the probability that problem can be solved by atleast one of them. 2. Three students A and B can solve the problem independently, the probabilities of solving the problem of each of them are 0.3, 0.6 respectively, then find the probability that problem can be solved by neither. 3. Three students X and Y can solve the problem independently, the probabilities of solving the problem of each of them are $\frac{1}{5} \& \frac{1}{10}$ respectively, then find the probability that problem can be solved.by only one of them.
2	Two events A and B are exclusive and $P(A) = 0.3\ and\ P(B) = 0.5$ then find $P(A/B)$
	Expected errors Students divide both probability $$P(A/B) = \frac{P(A)}{P(B)} = \frac{3}{5}$$ Independent events concept may influence exclusive events $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A) = 0.3$$ Correct answer: If A and B are exclusive then $P(A \cap B) = 0$ $$P(A/B) = \frac{P(A \cap B)}{P(B)} = 0$$

	Similar questions 1. Two events A and B are independent and $P(A) = 0.3\ and\ P(B) = 0.5$ then find $P(A/B)$. 2. Two events A and B are exclusive and exhaustive and $P(A) = 0.3\ and\ P(B) = 0.5$ then find $P(A \cup B/B)$. 3. Two events A and B are exclusive and $P(A) = 0.3\ and\ P(B) = 0.5$ then find $P(B/A)$.
3.	Three machines in a factory produce respectively 30%, 30% and 40% of items daily. The percentage of defective items of these machines are respectively 2, 2 and 1. An item is taken at random from the production and is found to be defective, find the probability that it is produced by machine A.
	Expected errors Students may use conditional probability in the place of bays theorem $$P(E/A) = 2\% = \frac{1}{50}$$ $$P(E/A) = \frac{2\%}{30\%} = \frac{1}{15}$$ Students may multiply direct percentages and do calculation mistakes $$P(A/E) = \frac{P(A)P(E/A)}{P(A)P(E/A) + P(B)P(E/B) + P(C)P(E/C)} = \frac{30\%}{30\% + 30\% + 40\%}$$ $$P(A/E) = \frac{P(A)P(E/A)}{P(A)P(E/A) + P(B)P(E/B) + P(C)P(E/C)} = \frac{2\%}{2\% + 2\% + 1\%}$$ Correct Answer $$P(A/E) = \frac{P(A)P(E/A)}{P(A)P(E/A) + P(B)P(E/B) + P(C)P(E/C)} = \frac{30\% * 2\%}{30\% * 2\% + 30\% * 2\% + 40\% * 1\%} = \frac{3}{8}$$

	Similar questions 1. In a factory 4000 units of an item are produced. There are three machines A, B and C in the factory. B and C produce equal number of items while machine A produces double items than B. The proportion of defective items produced by A, B and C are 2%, 1% and 3% respectively. An item is selected at random from the total production and it is found to be defective, find the probability that it comes from machine C. 2. Three machines in a factory produce respectively 20%, 50% and 30% of items daily. The percentage of defective items of these machines are respectively 3, 2 and 5. An item is taken at random from the production and is found to be defective, find the probability that it is produced by machine A.
4.	$$P(X) = \begin{cases} kX, & X = 0,1 \\ k(4-X), & X = 2,3 \\ kX^2, & X = 4 \end{cases}$$ For the above probability distribution of X find the value of k.
	Expexted errors Students will add all the probabilities without substituting the value of X. $$kX + k(4-X) + kX^2 = 1$$ $$kX + 4k - kX + kX^2 = 0$$ $$4k - kX^2 = 0$$ $$X^2 = 4$$ X= ± 2 Students commits error on error due to lack of practice. Correct asnswer $$k(0) + k(1) + k(2) + k(1) + k(16) = 1$$ $$k(20) = 1$$ $$k = \frac{1}{20}$$

	Similar questions 1. Let X denote the number of hours a class XII student studies during a randomly selected school day. The probability that X can take the values x_i , for an unknown constant 'k' $$P(X = x_i) = \begin{cases} 0.1, & x_i = 0 \\ kx_i, & x_i = 1,2 \\ k(5 - x_i), & x_i = 3,4 \end{cases}$$ (a) Find the value of k. (b) What is the probability that the student studied for at least two hours? Exactly two hours? At most two hours? 2. Let X denote the number of hours a class XII student studies during a randomly selected school day. The probability that X can take the values x_i , for an unknown constant 'k' $$P(X = x_i) = \begin{cases} 0.2, & x_i = 0 \\ kx_i, & x_i = 1,2 \\ k(5 - x_i), & x_i = 3 \\ 0, & otherwise \end{cases}$$ a) Find the value of k. b) What is the probability that the person watches two hours of television on a selected day? c) What is the probability that the person watches at least two hours of television on a selected day? d) What is the probability that the person watches at most 2 hours of television on a selected day?

Misconceptions in Probability along with examples:

Q.1) In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Solution: Let E1 and E2 be the respective events that the student knows the answer and he guesses the answer. Let A be the event that the answer is correct.

$P(E_1) = \frac{3}{4}$

$P(E_2) = \frac{1}{4}$

The probability that the student answered correctly, given that he knows the answer, is 1.

$\therefore$ $P(A|E_1) = 1$ (Error: Students mostly take probability as ¾)

REASON: Students think that Probability for guessing the correct answer is ¼ so The probability that the student answered correctly, given that he knows the answer is ¾

Probability that the student answered correctly, given that he guessed, is $\frac{1}{4}$.

$\therefore$ $P(A|E_2) = \frac{1}{4}$

The probability that the student knows the answer, given that he answered it correctly, is given by $P(E_1|A)$. By using Bayes' theorem, we obtain

$$P(E_1|A) = \frac{P(E1).P(A|E1)}{P(E1).P(A|E1)+P(E2).P(A|E2)}$$

$$= \frac{\frac{3}{4}.1}{\frac{3}{4}.1+\frac{1}{4}.\frac{1}{4}} \qquad \left(\frac{\frac{3}{4}.\frac{3}{4}}{\frac{3}{4}.\frac{3}{4}+\frac{1}{4}.\frac{1}{4}} = \frac{\frac{9}{16}}{\frac{10}{16}} = \frac{9}{10}\ due\ to\ wrong\ probabilty\ of\ P(A/E1\right)$$

$$= \frac{\frac{3}{4}}{\frac{3}{4}+\frac{1}{16}}$$

$$= \frac{\frac{3}{4}}{\frac{13}{16}}$$

$$= \frac{12}{13}$$

Q.2 There are three red balls and five blue balls in a bag. A student picks a ball at random. Find the probability that it is a red ball.

Sol. Number of red balls = 3

Number of blue balls = 5

P(A red ball)= $\frac{3}{5}$ (Error: Student does not take the total number of outcome in the sample space. He used only one part of the Sample space)

$Correct\ Answer = \frac{3}{8}$

Q.3 A and B throw a die alternatively till one of them gets a '6' and wins the game. Find the respective probabilities of winning , if A starts first.

Sol. Let S denote the success (getting a '6') and F denote the failure(not getting a '6')

Thus P(S)=$\frac{1}{6}$

& P(F)=$\frac{5}{6}$

P(A wins in the first throw)= P(S)=$\frac{1}{6}$

A gets the third throw, when the first throw by A and second throw by B result into a failures.

Therefore, P(A wins in the third throw)= P(FFS)=$\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

$= \left(\frac{5}{6}\right)^2 . \frac{1}{6}$ $\quad (Error: student\ multiplies\ as\ \frac{25}{216})$

(Reason: he is unable to identify the pattern involved)

P(A wins in the fifth throw)= P(FFFFS)=$\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

$=\left(\frac{5}{6}\right)^4 . \frac{1}{6}$ and so on

P(A wins)=$\frac{1}{6} + \left(\frac{5}{6}\right)^2 . \frac{1}{6} + \left(\frac{5}{6}\right)^4 . \frac{1}{6}$

$=\frac{\frac{1}{6}}{1-\frac{25}{36}}=\frac{6}{11}$ (Error: student does not apply the sum of infinite GP)

Q.4 A family had two children. It has a son, JOE. What is the probability that JOE's sibling is a brother?

Sol. The sample space S=$\{BB, BG, GB, GG\}$ Define, E: Family has a son=$\{BB, BG, GB\}$

F: Family has two sons =$\{BB\}$

$P(F/E)=\frac{P(F\cap E)}{P(E)}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}$ (Error: JOE's sibling is equally likely to have been born male or female suggests that the probability of other child is a boy is ½)

Q.5 A bag contains 5 red marbles and 3 black marbles. Three marbles are drawn one by one without replacement. What is the probability that at least one of the three marbles drawn be black, if the first marble is red?

Sol. Let R and B represent 5 red marbles and

3 black marbles, respectively. For at least one of the three marbles drawn be black, if the first marble is red, the following three conditions will be followed :

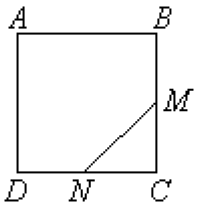
(i) Second ball is black and third is red = E1
(ii) Second ball is black and third is also black=E2
(iii) Second ball is red and third is black=E3

$\therefore$ P(E1)=$\frac{5}{8}.\frac{3}{7}.\frac{4}{6}$ (Error: Student does like $\frac{5}{8}+\frac{3}{7}+\frac{4}{6}$)

$=\frac{60}{336}=\frac{5}{28}$

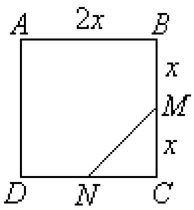
Non-routine questions

Q1. *ABCD* is a square. *M* is the midpoint of *BC* and *N* is the midpoint of *CD*. A point is selected at random in the square. Calculate the probability that it lies in the triangle *MCN*.



Solution:

Let $2x$ be the length of the square.



Area of square = $2x \times 2x = 4x^2$

Area of triangle *MCN* is $\frac{1}{2}x^2$

$$\begin{aligned} \text{P(point in the triangle)} &= \frac{1}{2}x^2 \div 4x^2 \\ &= \frac{1}{2}x^2 \times \frac{1}{4x^2} \\ &= \frac{1}{8} \end{aligned}$$

Q2. At a car park there are 100 vehicles, 60 of which are cars, 30 are vans and the remainder are lorries. If every vehicle is equally likely to leave, find the probability of:

a) van leaving first.
b) car leaving second if either a lorry or van had left first.

Solution:

a) Let *S* be the sample space and *A* be the event of a van leaving first.

n(*S*) = 100
n(*A*) = 30

Probability of a van leaving first=$\frac{30}{100} = \frac{3}{10}$

b) If either a lorry or van had left first, then there would be 99 vehicles remaining, 60 of which are cars. Let *T* be the sample space and *C* be the event of a car leaving.

n(*T*) = 99
n(*C*) = 60

Probability of a car leaving after a lorry or van has left:=$\frac{60}{99}$=$\frac{20}{33}$

Q.3 A letter is known to have come either from 'TATA NAGAR' or from 'CALCUTTA'. On the envelope, just two consecutive letters TA are visible. What is the probability that the Probability that the letter came from 'TATA NAGAR'

Sol. Let

E_1 be the event that a letter is from TATA NAGAR
E2 be the event that letter is from CALCUTTA. And E3 be the event that two consecutive letters TA are visible on envelope
$\therefore$ P(E1)=$\frac{1}{2}$
And P(E2)= $\frac{1}{2}$ Also,
if letter is from TATA NAGAR, we see that the events of two consecutive letters visible
are {TA, AT, TA, AN, NA, AG, GA, AR}
And if the letter is from CALCUTTA, we see that the events of getting two consecutive letters visible are {CA, AL, LC, CU, UT, TT, TA}.
Thus,
P(E3/E1)=$\frac{2}{8}$ $and\ P\left(\frac{E2}{E1}\right) = \frac{1}{7}$

So, P(E1/E3) $= \frac{P(E1).P(E3/E2)}{P(E1).P(E3/E2)+P(E2).P(E3/E2)}$ $= \frac{\frac{1}{2}.\frac{2}{8}}{\frac{1}{2}.\frac{2}{8}+\frac{1}{2}.\frac{1}{7}} = \frac{\frac{1}{8}}{\frac{1}{8}+\frac{1}{14}} = \frac{7}{11}$

Q4. A bag contains (2n+ 1) coins. It is known that n of these coins have a head on both sides whereas the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that, the toss results in a head is $\frac{31}{42}$ then determine the value of n

Sol. Given, n coins have head on both sides and (n + 1) coins are fair coins.

Let E1= Event that an unfair coin is selected

E2= Event that a fair coin is selected

E= Event that the toss results in a head
So, P(E1)=$\frac{n}{2n+1}$ and P(E2)=$\frac{n+1}{2n+1}$

And P(E/E1)=1 and P(E/E2)=$\frac{1}{2}$

P(E)= P(E1).P(E/E1)+ P(E2).P(E/E2)

$$\frac{31}{42} = \frac{n}{2n+1}.1+\frac{n+1}{2n+1}.\frac{1}{2}$$

Simplifying we get $n = 10$

Q.5 The probability that at least one of the two events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, evaluate $P(A') + P(B')$

Sol. We know that, A∪B denotes the occurrence of at least one of A and B and A∩B denotes the occurrence of both A and B, simultaneously.

Thus, from the question we have:
$P(A \cup B) = 0.6\ and\ P(A \cap B) = 0.3$
$And\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$0.6 = P(A) + P(B) - 0.3$$

$$P(A) + P(B) = 0.9$$

$$1 - P(A') + 1 - P(B') = 0.9$$

$$So\ P(A') + P(B') = 1.1$$

COMMON MISTAKES AND REMEDIES

S.NO.	COMMON MISTAKES	REMEDIES
1.	Confusion in independent and Mutually exclusive events	Independent events have common outcome and probability of one does not affect probability of other events .then P$(A \cap B) = P(A).P(B)$ Mutually exclusive events have no outcomes. then $P(A \cup B) = P(A) + P(B)$
2.	Difference in dependent and independent events	Total outcomes for $2^{nd}, 3^{rd}$ ....out comes reduce by one. Ex:- two kings are drawn one by one from a deck of 52 cards. $P(K_1)$ =$\frac{4}{52}$ and P(K_2) =3/51 Total outcomes for ,$2^{nd} 3^{rd}$ ....out comes will remain same. Ex:- two kings are drawn one by one from a deck of 52 cards. P(K_1) =4/52 and P(K_2) =4/52
3	Conditional probability	$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)} = \frac{n(A\cap B)}{n(B)}$ i.e. selection of the element of A from B
4	Students are un able to understand the question based on Bays Theorem	Formula :-$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E1\ \ )P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)+\cdots\ldots\ldots+P(E_n)P\left(\frac{A}{E_n}\right)}$ $=\frac{P(E_1\cap A)}{P(A)}$. selection of the element of E_1 from A. In question highlight on the last sentence . Three urns A ,B and C contain 6 red and 4 white ball; 2 red and 6 white balls,1 red and 5 white balls respectively .An urn is chosen at random and a ball is drawn is found to be red, find the probability that the ball was drawn from urn A .

5	Students are un able to understand the question based on Binomial distribution.	Formula $(P+q)^n$ The probability distribution is given by P$(X=r)=n_{C_r}q^{n-r}p^r$ 1.trials are independent . 2. each trial has exactly two outcomes. 3.there should be finite number of trials.

1.In question , with replacement is mentioned , then it is independent event.
2. In question , without replacement is mentioned , then it is dependent event.
3. if number of trials are more than three and the trial has only two outcomes ,then is the question of binomial distribution.